A Double Arity Hierarchy Theorem for Transitive Closure Logic

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Abstract

In this paper we prove that the k–ary fragment of transitive closure logic is not contained in the extension of the $(k-1)$–ary fragment of partial fixed point logic by all $(2k-1)$–ary generalized quantifiers. As a consequence, the arity hierarchies of all the familiar forms of fixed point logic are strict simultaneously with respect to the arity of the induction predicates and the arity of generalized quantifiers.

Although it is known that our theorem cannot be extended to the sublogic deterministic transitive closure logic, we show that an extension is possible when we close this logic under congruence.

1 Introduction

Due to the well known limitations of the expressive power of first–order logic on finite structures, finite model theory mainly deals with extensions of first–order logic. Many of the operators used to form these extensions have a natural notion of an *arity*. For example, the arity of a fixed point operator is the arity of the relation defined by the fixed point. The fragments allowing only operators of a bounded arity form a natural hierarchy inside of such logics, and the obvious question is whether this hierarchy is strict. An affirmative answer to this question for various extensions of first–order logic by fixed point operators and transitive closure operators has been given in [5].

A different concept is to extend first–order logic by *Lindström quantifiers*. Again, such quantifiers have a natural notion of an arity which was studied in [7, 8]. Among other things it was proved there that for each k there exists a query which is definable in fixed point logic, but not in any extension of first–order logic by Lindström quantifiers of arity $< k$.

The main theorem of this paper combines these results. Denoting, for any $k \geq 0$, the k–ary fragment of *transitive closure logic* by TC^k, the k–ary fragment

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of *simultaneous partial fixed point logic* by $\mathsf{s\text{–}PFP}^k$, and the extension of a logic L by all k–ary Lindström quantifiers by $\mathsf{L}(\mathbf{Q}^k)$, it reads:

Hierachy Theorem 1.1 *For each* $k \geq 1$ *we have* $\mathsf{TC}^k \not\subseteq \mathsf{s\text{–}PFP}^{k-1}(\mathbf{Q}^{2k-1})$.

Although not the most familiar, simultaneous partial fixed point logic (the extension of partial fixed point logic allowing simultaneous inductions) is the most expressive of the fixed point logics usually being studied in finite model theory. Thus our formulation of the theorem using this logic is the most general. It implies the same result for more common fixed point logics, such as for example *least fixed point logic*.

Our theorem is optimal in two ways: TC^k is contained in both $\mathsf{s\text{–}PFP}^k$ and $\mathsf{FO}(\mathbf{Q}^{2k})$ (the extension of first–order logic by all $2k$–ary Lindström quantifiers).

Furthermore, it should be remarked that the theorem does not hold for a uniform (finite) signature, i.e. there is no finite signature σ such that for each $k \geq 1$ there is a TC^k–definable query which is not definable in $\mathsf{s\text{–}PFP}^{k-1}(\mathbf{Q}^{2k-1})$. The reason for this is that all queries of a signature which contains at most k–ary relation symbols can be defined in $\mathsf{FO}(\mathbf{Q}^k)$.

An important sublogic of transitive closure logic is *deterministic transitive closure logic* DTC. It was shown in [5] that $\mathsf{DTC} \subseteq \mathsf{s\text{–}PFP}^1$. Thus our hierarchy theorem does not extend to this logic. A reason for this can be seen in the fact that DTC is not congruence closed. In Section 4 we are going to investigate the *vectorized congruence closure* of DTC and show that on the one hand the hierarchy theorem extends to this logic, whereas on the other hand it inherits one of the most interesting features of DTC, namely to capture the complexity class $\mathsf{Logspace}$ on ordered finite structures.

Although the logics we are dealing with are mainly of interest in finite model theory, we need not restrict ourselves to the class of finite structures. Rather it should be remarked that all our results hold in particular on that class.

2 Preliminaries

Our notation is standard, as it can for example be found in [2]. We often abbreviate tuples $x_1 \ldots x_k$ by $\overset{k}{x}$ or $\bar{x}$. Recall that an interpretation is a pair $(\mathfrak{A}, \alpha)$, where $\mathfrak{A}$ is a structure over an appropriate signature and α assigns a value in $\mathfrak{A}$ to each variable. For convenience we assume that signatures do not contain any function or constant symbols.

2.1 Transitive closure logic

The class TC of *transitive closure formulae* is given by the usual rules for building first–order formulae and the new rule:

$$\text{(TC)} \quad \frac{\varphi}{[\mathrm{TC}_{\overset{k}{x},\overset{k}{y}}\ \varphi]\overset{k}{u},\overset{k}{v}} \quad \text{where } k \geq 1 \text{ and } \overset{k}{u} \text{ and } \overset{k}{v} \text{ are } k\text{–tuples of terms.}$$

$[\mathrm{TC}_{\overset{k}{x},\overset{k}{y}} \ldots]$ is called a *k–ary transitive closure operator*.

To define the semantics, let $\mathfrak{I} = (\mathfrak{A}, \alpha)$ be an interpretation for a transitive closure formula φ. Then

$$\mathfrak{I} \models [\mathrm{TC}_{\overset{k}{x},\overset{k}{y}}\ \varphi]\overset{k}{u},\overset{k}{v}$$

if and only if there exist $n \geq 2, \overset{k}{a}_1, \ldots, \overset{k}{a}_n \in A$ such that $\alpha(\overset{k}{u}) = \overset{k}{a}_1, \alpha(\overset{k}{v}) = \overset{k}{a}_n$ and $\mathfrak{I} \models \varphi[\overset{k}{a}_i, \overset{k}{a}_{i+1}]$ for each $i < n$.

Now the semantics of the logic $\mathsf{TC} = (\mathrm{TC}, \models)$ can be defined inductively with respect to the calculus given above.

Note that we do not require the transitive closure to be reflexive.

A TC–formula is *k–ary* if it contains at most k–ary TC–operators. The sublogic of TC whose formulae are the k–ary TC–formulae is denoted by TC^k. We usually refer to TC^k as the *k–ary fragment of* TC.

2.2 Partial fixed point logic

As mentioned in the introduction we are going to work with a generalization of the common partial fixed point logic which allows simultaneous inductions. It is known that this does not increase the expressive power in general, but the situation is different if we restrict the arity of the formulae.

The class s–FP of *simultaneous fixed point formulae* is given by means of the calculus consisting of the first–order rules and the rule

$$\text{(s–FP)} \quad \frac{\varphi_1, \ldots, \varphi_m}{[\mathrm{FP}_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m}\ \varphi_1, \ldots, \varphi_m]\,\bar{u}}$$

where $m \geq 1$ and, for each $i \leq m$, X_i is a relation variable whose arity matches the length of $\bar{x}_i$, and $\bar{u}$ is a tuple of terms of the same length as $\bar{x}_1$.

To define the semantics, for each simultaneous fixed point formula

$$[\mathrm{FP}_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m}\ \varphi_1, \ldots, \varphi_m]\,\bar{u}$$

and interpretation $\mathfrak{I} = (\mathfrak{A}, \alpha)$ we define sequences $(X^{\mathfrak{I}}_{ji})_{i \geq 0}$ $(1 \leq j \leq m)$ of relations on A by

$$
\begin{aligned}
X^{\mathfrak{I}}_{j0} &:= \emptyset \\
X^{\mathfrak{I}}_{j(i+1)} &:= \{\bar{a} \in A \mid \mathfrak{I} \models \varphi_j[\bar{a}, X^{\mathfrak{I}}_{1i}, \ldots, X^{\mathfrak{I}}_{mi}]\} \\
X^{\mathfrak{I}}_{j\infty} &:= \begin{cases} X^{\mathfrak{I}}_{jk} & \text{where } k = min\{i \mid \bigwedge_{j'=1}^{m} X^{\mathfrak{I}}_{j'i} = X^{\mathfrak{I}}_{j'(i+1)}\} \\ & \text{(if such a } k \text{ exists)} \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}
$$

We let

$$\mathfrak{I} \models [\mathrm{FP}_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m}\ \varphi_1, \ldots, \varphi_m]\, \bar{u} \Longleftrightarrow \alpha(\bar{u}) \in X^{\mathfrak{I}}_{1\infty}$$

and define the semantics of the *simultaneous partial fixed point logic* $\mathsf{s\text{–}PFP} = (\text{s–FP}, \models)$ inductively.

FP is the subclass of s–FP built by the first–order rules and the restriction of the (s–FP)–rule to the case $m = 1$. The corresponding sublogic *partial fixed point logic* $\mathsf{PFP} = (\text{FP}, \models)$ is known to have already the same expressive power as $\mathsf{s\text{–}PFP}$.

A simultaneous fixed point operator $[\text{s–FP}_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m} \ldots]$ is *k–ary* if k is the maximum of the arities of $X_1, \ldots, X_m$. A (simultaneous) fixed point formula is *k–ary* if it contains at most k–ary (simultaneous) fixed point operators. $\mathsf{(s)\text{–}PFP}^k$ denotes the k–ary fragment of $\mathsf{(s)\text{–}PFP}$.

Remark 2.1 *There are several other fixed point logics being studied. Since the expressive power of the k–ary fragments of most of these logics is somewhere between TC^k and $\mathsf{s\text{–}PFP}^k$ our hierarchy theorem implies similar statements for these logics (cf. [4] for more details).*

2.3 Lindström logics

In the following we let $\mathsf{L} = (\mathrm{L}, \models)$ be a logic (where L is a class of formulas and $\models$ a satisfaction relation associating with each formula a relation on each structure of the same signature), $\sigma = \{R_1, \ldots, R_s\}$ a signature consisting of l_i–ary relation symbols R_i, and $\mathcal{C}$ a class of σ–structures which is closed under isomorphism.

The class $\mathrm{L}(Q_{\mathcal{C}})$ is the closure of L under the rule

$$(Q_{\mathcal{C}}) \quad \frac{\varphi_1(\overset{l_1}{x_1}), \ldots, \varphi_m(\overset{l_m}{x_m})}{\left[Q_{\mathcal{C}} \overset{l_1}{x_1}, \ldots, \overset{l_m}{x_m}\ \varphi_1(\overset{l_1}{x_1}), \ldots, \varphi_m(\overset{l_m}{x_m})\right]}$$

and the rules used to build L. (To make this precise we would need an exact definition of the notion "formula". However, it is clear what is meant here if L is one of the logics we have defined in this paper so far.)

$Q_{\mathcal{C}}$ is called the *Lindström–quantifier* associated with $\mathcal{C}$.

Let $\mathfrak{I} = (\mathfrak{A}, \beta)$ be an interpretation. For each formula $\varphi(\overset{k}{x})$ we let

$$\varphi(\overset{k}{x})^{\mathfrak{I}} := \left\{ \overset{k}{a} \in A^k \mid \mathfrak{I} \models \varphi[\overset{k}{a}] \right\}.$$

As usual, the semantics of the logic $\mathsf{L}(Q_{\mathcal{C}}) = (L(Q_{\mathcal{C}}), \models)$ is defined inductively, the only interesting case being:

$$\mathfrak{I} \models \left[Q_{\mathcal{C}} \overset{l_1}{x_1}, \ldots, \overset{l_m}{x_m}\ \varphi_1(\overset{l_1}{x_1}), \ldots, \varphi_m(\overset{l_m}{x_m}) \right]$$

iff the σ–structure $\left(A, \varphi_1(\overset{l_1}{x_1})^{\mathfrak{I}}, \ldots, \varphi_s(\overset{l_m}{x_m})^{\mathfrak{I}}\right)$ belongs to $\mathcal{C}$.

Similarly we define the logics $\mathsf{L}(\mathbf{Q})$ for sets $\mathbf{Q}$ of Lindström-quantifiers.

The arity of the Lindström–quantifier $Q_{\mathcal{C}}$ associated with a class $\mathcal{C}$ of σ–structures is

$$\max\Big(\{l \mid \sigma \text{ contains an } l\text{–ary relation symbol}\} \cup \{0\}\Big)$$

For each $l \geq 1$ we let

$$\mathbf{Q}^l := \{Q_{\mathcal{C}} \mid Q_{\mathcal{C}} \text{ is a Lindström-quantifier of arity } \leq l\}\ .$$

2.4 Vectorized Lindström logics

For a relational signature σ we let σ^k consist of a new rk–ary relation symbol R^k for each r–ary $R \in \sigma$.

The *k–ary version* of a class $\mathcal{C}$ of σ–structures is the class $\mathcal{C}^k$ of σ^k–structures defined as follows: A σ^k–structure $\mathfrak{B}$ belongs to $\mathcal{C}^k$ iff the structure $(B^k, ((R^k)^{\mathfrak{B}})_{R\in\sigma})$ belongs to $\mathcal{C}$. Here we consider $(R^k)^{\mathfrak{B}}$ as an r–ary relation on B^k rather than an rk–ary relation on B (for any r–ary $R \in \sigma$).

Note that $\mathcal{C}^1 = \mathcal{C}$.

The *vectorization* of a Lindström logic $\mathsf{L}(\mathbf{Q})$ is defined to be

$$\mathsf{L}^{\omega}(\mathbf{Q}) := \mathsf{L}\left(\{Q_{\mathcal{C}^k} \mid Q_{\mathcal{C}} \in \mathbf{Q}, k \geq 1\}\right).$$

Finally, the k–ary fragment of $\mathsf{L}^{\omega}(\mathbf{Q})$ is the logic

$$\mathsf{L}^{k}(\mathbf{Q}) := \mathsf{L}\left(\{Q_{\mathcal{C}^l} \mid Q_{\mathcal{C}} \in \mathbf{Q}, 1 \leq l \leq k\}\right).$$

The vectorization of a Lindström quantifier is a quite natural concept which often occurs in finite model theory. For example, TC can be considered as a vectorized Lindström logic. Let $\sigma_T = \{E, L, R\}$ (E binary, L, R unary) and

$$\mathcal{TC} := \Big\{\mathfrak{A} \mid \mathfrak{A} \text{ is a } \sigma_T\text{–structure}, \exists a \in L^{\mathfrak{A}}, b \in R^{\mathfrak{A}} : \mathfrak{A} \models [\mathrm{TC}_{x,y} Exy]a, b\Big\}$$

(Thus $\mathcal{TC}$ is the class of σ_T–structures that have an E–path from the Left to the Right.) It is not hard to see that for each $k \geq 1$ we have $\mathsf{TC}^k = \mathsf{FO}^k(Q_{\mathcal{TC}})$. Since the quantifier $Q_{\mathcal{TC}^k}$ is $2k$–ary this implies $\mathsf{TC}^k \subseteq \mathsf{FO}(\mathbf{Q}^{2k})$.

2.5 An Ehrenfeucht-Fraïssé game

In this subsection we are going to define an Ehrenfeucht-Fraïssé game that is sufficient for the logics $\mathsf{s–PFP}^k(\mathbf{Q}^l)$ ("sufficiency" meaning that if the duplicator has a winning strategy for the game on two structures they cannot be distinguished in the logic).

The game merges the k–ary r–pebble game (corresponding to the logic $\mathsf{s–PFP}^k$) introduced in [4, 5] with the l–bijective pebble game of length r (corresponding to the logic $\mathsf{FO}(\mathbf{Q}^l)$) introduced in [7, 8].

Definition 2.2 *Let $k, l, r \geq 1$. The* k-ary l-bijective r-pebble game $AB_r^{k,l}(\mathfrak{A}, \mathfrak{B})$ *on a pair $\mathfrak{A}, \mathfrak{B}$ of structures of the same signature is played by two players, the challenger and the duplicator, with $2r$ pebbles $P_1, Q_1, \ldots, P_r, Q_r$.*

We say that a pair (P_i, Q_i) of pebbles is on the board *in a situation of the game if it is placed on the structures in that situation. The other pebbles are called* free.

Each situation of the game and each pebble on the board have a depth $d \geq 0$. The game starts in the situation with depth 0 and all pebbles free.

If $\mathfrak{A}$ and $\mathfrak{B}$ are of the same cardinality, in each round of the game the challenger selects one of the following moves:

Q-move *: The duplicator selects a bijection $f : A \longrightarrow B$.*

The challenger places $m \leq l$ free pebbles $P_{i_1}, \ldots, P_{i_m}$ on $a_1, \ldots, a_m \in A$ and the corresponding pebbles $Q_{i_1}, \ldots, Q_{i_m}$ on $f(a_1), \ldots, f(a_m) \in B$.

The depth of the P_{i_j} and Q_{i_j} is defined to be the current depth of the game.

I-move *: The depth of the game is increased by 1.*

R-move *: The depth of the game is reduced to a $d \geq 1$ less than or equal to the current depth. The challenger selects $j \leq k$ pairs $(P_{i_1}, Q_{i_1}), \ldots, (P_{i_j}, Q_{i_j})$ of pebbles of depth $\geq d$ to be left on the board. Their depth is reduced to d. All other pebbles of depth $\geq d$ are removed from the structures.*

The duplicator wins the game if in each situation the pairs of elements which are pebbled by corresponding pebbles P_i, Q_i form a partial isomorphism between

$\mathfrak{A}$ *and* $\mathfrak{B}$. *Otherwise, or if* $\mathfrak{A}$ *and* $\mathfrak{B}$ *do not have the same cardinality, the challenger wins the game.*

To relate the game to the logics s–PFP$^k(\mathbf{Q}^l)$ ($k, l \geq 0$) we need a notion of quantifier–rank for these logics. It is defined inductively in the usual way, adding the clauses

$$\mathrm{qr}([\text{s–FP}_{\overset{k_1}{x_1},X_1,\ldots,\overset{k_m}{x_m},X_m}\ \varphi_1,\ldots,\varphi_m]\overset{k_1}{u}) = \max\{\mathrm{qr}(\varphi_i)+k_i \mid i \leq m\}\ .$$

and

$$\mathrm{qr}([Q_{\mathcal{C}}\overset{l_1}{x_1},\ldots,\overset{l_m}{x_m}\ \varphi_1,\ldots,\varphi_m]) = \max\{\mathrm{qr}(\varphi_i)+l_i \mid i \leq m\}$$

(where $\mathcal{C}$ is a class of signature $\{R_1,\ldots,R_m\}$ with l_i–ary R_i).

A proof of the following theorem, which is a straightforward generalization of the proofs of the corresponding theorems in [5] and [8], can be found in [4].

Theorem 2.3 *Let* $k, l, r \geq 1$ *and* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two structures of the same signature such that the duplicator has a winning strategy for the* k*–ary* l*–bijective* r*–pebble game on* $\mathfrak{A}$ *and* $\mathfrak{B}$.

Then the same s–PFP$^k(\mathbf{Q}^l)$*–sentences of quantifier rank* $\leq r$ *hold in* $\mathfrak{A}$ *and* $\mathfrak{B}$.

3 Proof of the hierarchy theorem

Let V be a k–ary and E a $2k$–ary relation symbol. We start by defining for each positive integer n a $\{V,E\}$-structure $\mathfrak{C}_n = (C_n, V^{\mathfrak{C}_n}, E^{\mathfrak{C}_n})$.

Definition 3.1 *The universe of* $\mathfrak{C}_n$ *is the cartesian product* $C_n = \{1,\ldots,n\} \times \{1,\ldots,k\} \times \{1,2\}$.

For each $1 \leq i \leq n$*, the* i*th row of* C_n *is* $\mathrm{Row}_i = \{i\} \times \{1,\ldots,k\} \times \{1,2\}$.

For each $1 \leq j \leq k$*, the* j*th column of* C_n *is* $\mathrm{Col}_j = \{1,\ldots,n\} \times \{j\} \times \{1,2\}$.

The left side of C_n *is* $L = \{1,\ldots,n\} \times \{1,\ldots,k\} \times \{1\}$.

The interpretation of the relation symbol V *is*

$$V^{\mathfrak{C}_n} = \bigcup_{1 \leq i \leq n} ((\mathrm{Col}_1 \times \cdots \times \mathrm{Col}_k) \cap (\mathrm{Row}_i)^k).$$

Finally, $E^{\mathfrak{C}_n}$ *consists of all tuples* $\overset{k}{a}\overset{k}{b} \in (V^{\mathfrak{C}_n})^2$ *such that*

- *for some* $1 \leq i < n$, $\overset{k}{a} \in (\mathrm{Row}_i)^k$ *and* $\overset{k}{b} \in (\mathrm{Row}_{i+1})^k$, *and*
- $\mathrm{card}(\{a_1,\ldots,a_k,b_1,\ldots,b_k\} \cap L)$ *is even.*

(By $\mathrm{card}(M)$ *we denote the cardinality of a set* M*.)*

The relation $V^{\mathfrak{C}_n}$ consists of all k-tuples that are contained in one row but intersect all columns. On these tuples we define a directed graph with edge relation $E^{\mathfrak{C}_n}$. We can divide this directed graph into two connected components,

$$\text{Even} = \{(a_1, \ldots, a_k) \in V^{\mathfrak{C}_n} \mid \text{card}(\{a_1, \ldots, a_k\} \cap L) \text{ is even}\}, \text{ and}$$

$$\text{Odd} = \{(a_1, \ldots, a_k) \in V^{\mathfrak{C}_n} \mid \text{card}(\{a_1, \ldots, a_k\} \cap L) \text{ is odd}\},$$

both of which are isomorphic to the product of the successor relation on n and the complete graph on 2^{k-1} vertices.

Let us consider now the $(k-1)$-ary $(2k-1)$-bijective r-pebble game on $\mathfrak{C}_n$, $AB_r^{k-1,2k-1}(\mathfrak{C}_n, \mathfrak{C}_n)$. Of course, the duplicator wins this game if he is allowed to use the identity function id_{C_n} (or some other automorphism of $\mathfrak{C}_n$). Our aim is to show that for large enough n, the duplicator has — in addition to such trivial winning strategies — a winning strategy consisting of bijections which do not preserve the relation $E^{\mathfrak{C}_n}$.

Definition 3.2 *(1) For each set $K \subseteq L$, let g_K be the bijection $C_n \to C_n$ which swaps the elements $(i,j,1)$ and $(i,j,2)$ for all $(i,j,1) \in K$, and is the identity elsewhere; i.e.,*

$$g_K(a) = \begin{cases} (i,j,2), & \text{if } a = (i,j,1) \in K \\ (i,j,1), & \text{if } a = (i,j,2) \text{ and } (i,j,1) \in K \\ a, & \text{otherwise.} \end{cases}$$

(2) Let $K \subseteq L$ and $1 \le i \le n$. The bijection g_K is even *(*odd*, resp.) at i if $\text{card}(K \cap \text{Row}_i)$ is even (odd, resp.). Furthermore, g_K is* twisted *between i and $i+1$ if it is even at i and odd at $i+1$, or vice versa.*

(3) For each $1 \le i \le n$, let $\mathcal{G}_n^i$ be the set of all those bijections g_K that are even at all $i' \le i$ and odd at all $i' > i$. We denote by $\mathcal{G}_n$ the union $\bigcup_{1 \le i < n} \mathcal{G}_n^i$ of the sets $\mathcal{G}_n^i$.

Clearly the bijection g_K preserves the relation $V^{\mathfrak{C}_n}$ for any $K \subseteq L$. Note further that g_K is odd at i if and only if it swaps the even and odd tuples of Row_i: for every $\overset{k}{a} \in V^{\mathfrak{C}_n} \cap (\text{Row}_i)^k$, $\overset{k}{a} \in \text{Even} \iff g_K(\overset{k}{a}) \in \text{Odd}$. Hence, the restriction of g_K to a set $A \subseteq C_n$, $g_K{\restriction}A$, is a partial automorphism of $\mathfrak{C}_n$, unless there is an i such that g_K is twisted between i and $i+1$ and $A^k \cap V^{\mathfrak{C}_n} \cap (\text{Row}_i)^k$ and $A^k \cap V^{\mathfrak{C}_n} \cap (\text{Row}_{i+1})^k$ are both non-empty. In particular, the bijections in $\mathcal{G}_n^i$ are not automorphisms of $\mathfrak{C}_n$, but they preserve the relation $E^{\mathfrak{C}_n}$ everywhere, except between rows Row_i and Row_{i+1}.

Proposition 3.3 *Let $G_r(\mathfrak{C}_n)$ be the game with the same rules as the game $AB_r^{k-1,2k-1}(\mathfrak{C}_n, \mathfrak{C}_n)$, except that the duplicator is only allowed to move bijections that are in $\mathcal{G}_n$. If $n = 2^r$, then the duplicator has a winning strategy in the game $G_r(\mathfrak{C}_n)$.*

In proving Proposition 3.3 we will use the following simple observations. For all $s < t \leq n$ let $I(s,t)$ be the interval of C_n consisting of all the rows between Row_{s+1} and Row_t; i.e., $I(s,t) = \bigcup_{s<i\leq t} \text{Row}_i$. We denote the complement $C_n \setminus I(s,t)$ by $\overline{I}(s,t)$. Clearly $\mathfrak{C}_n \restriction I(s,t)$ (the restriction of the structure $\mathfrak{C}_n$ to the set $I(s,t)$) is isomorphic to $\mathfrak{C}_m$, where $m = t - s$; indeed, the mapping $f : \mathfrak{C}_m \to \mathfrak{C}_n \restriction I(s,t)$, $(i,j,l) \mapsto (s+i,j,l)$ is an isomorphism. Furthermore, for any fixed $g_K \in \mathcal{G}_n^t$, f induces a one-to-one mapping

$$\mathcal{F} : \mathcal{G}_m \longrightarrow \{g_J \in \mathcal{G}_n \mid g_J \restriction \overline{I}(s,t) = g_K \restriction \overline{I}(s,t)\}$$

defined by

$$\mathcal{F}(g_H) = g_J, \quad \text{where} \quad J = f[H] \cup (K \cap \overline{I}(s,t))$$

(denoting $\{f(h) \mid h \in H\}$ by $f[H]$). Note that $g_H \in \mathcal{G}_m^i$ implies that $\mathcal{F}(g_H) \in \mathcal{G}_n^{s+i}$, and so $\mathcal{F}(g_H) \restriction (f[A] \cup \overline{I}(s,t))$ is a partial automorphism of $\mathfrak{C}_n$ whenever $g_H \restriction A$ is a partial automorphism of $\mathfrak{C}_m$. In particular, $\mathcal{F}$ lifts any winning strategy of the duplicator in $G_r(\mathfrak{C}_m)$ to a winning strategy in $G_r(\mathfrak{C}_n)$.

Hence we have:

Lemma 3.4 *Assume that $s < t \leq n$, $m = t - s$, the duplicator has a winning strategy in $G_r(\mathfrak{C}_m)$, and $g_K \in \mathcal{G}_n^t$. Then the duplicator has a winning strategy in $G_r(\mathfrak{C}_n)$ which consists of bijections g_J such that $g_J \restriction \overline{I}(s,t) = g_K \restriction \overline{I}(s,t)$.*

Proof (of Proposition 3.3). We prove the claim by induction on r. Note first that the duplicator simply cannot loose in $G_1(\mathfrak{C}_2)$, since $f \restriction \{a\}$ is a partial automorphism of $\mathfrak{C}_n$ for any bijection $f : C_2 \to C_2$ and any $a \in C_2$.

Assume then that the claim holds for all $r' < r$, and let $n = 2^r$. We shall now describe a winning strategy for the duplicator in $G_r(\mathfrak{C}_n)$.

Let us first suppose that the challenger decides to start the game with a Q–move. Choose any bijection $g_K \in \mathcal{G}_n^i$, for $i = 2^{r-1} = n/2$, as the first move of the duplicator. The challenger answers by putting some pebbles, say $P_1, \ldots, P_l$, on some elements $a_1, \ldots, a_l \in C_n$, and the corresponding pebbles $Q_1, \ldots, Q_l$ on the elements $g_K(a_1), \ldots, g_K(a_l) \in C_n$, where $l < 2k$. The next move of the duplicator depends now on how the pebbled elements are distributed between the first half $F = I(0, n/2)$ and the second half $S = I(n/2, n)$ of C_n. By the obvious symmetry, it suffices to consider here the case $\text{card}(A \cap F) \leq \text{card}(A \cap S)$, where $A = \{a_1, \ldots, a_l\}$.

Thus, we assume that F contains at most $l/2$ of the pebbled elements. Let $r' = r - l$ and $n' = 2^{r'}$. Since F can be divided into $2^{l-1} > l/2$ mutually disjoint intervals of length n', there exists an interval $I(s,t) \subseteq F$ such that $t - s = n'$ and $I(s,t) \cap A = \emptyset$. Choose then for each $t < i \leq n/2$ an element $c_i \in L \cap \text{Row}_i$ such that c_i is not in the same column and row with any element of A; this is possible because every row in the first half F contains at most $l/2 < k$ elements of A. Let $I = \{c_i \mid t < i \leq n/2\}$, and let J be the symmetric difference of the sets K and I. Thus, $g_J \in \mathcal{G}_n^t$ (since $g_K \in \mathcal{G}_n^{\frac{n}{2}}$), and g_J agrees with g_K on the set A, i.e., $g_J \restriction A = g_K \restriction A$.

By the induction hypothesis, the duplicator has a winning strategy in $G_{r'}(\mathfrak{C}_{n'})$. Hence, by Lemma 3.4, he has a winning strategy σ in $G_{r'}(\mathfrak{C}_n)$ consisting of bijections g_H such that $g_H \restriction \overline{I}(s,t) = g_J \restriction \overline{I}(s,t)$. In particular, all the bijections given by σ agree with g_K on the set A, whence σ remains a winning strategy even if the already pebbled elements $a_1, \ldots, a_l$ and $g_K(a_1), \ldots, g_K(a_l)$ are taken into account.

Thus, the duplicator can play according to σ in the continuation of the game $G_r(\mathfrak{C}_n)$ without loosing as long as there are at most r' pairs of free pebbles. However, since the game started with a Q–move the pebbles $P_1, \ldots, P_l$ and $Q_1, \ldots, Q_l$ are of depth 0 and thus can never be removed.

Suppose next that the game starts with an I–move. Since it does not make much sense for the challenger to make two succeeding I–moves or an R–move immediately after an I–move, the second move will be a Q–move. The duplicator uses the same strategy for this Q–move as described above (for the first move).

However, this time the pebbles $P_1, \ldots, P_l$ and $Q_1, \ldots, Q_l$ are of depth 1. Thus it may happen that some of them are removed in an R–move and there are more than r' free pebble pairs.

Since there are no pebbles of lower depth than $P_1, \ldots, P_l$, the total number l' of pebble pairs on board after this R-move is at most $k-1$. Let g_M be the previous bijection of the duplicator, and let B be the set of the l' elements with P-pebbles on them. Repeating now the argument above we find a new interval $I(u,v) \subseteq C_n$ of length $n'' = v - u = 2^{r-l'} = 2^{r''}$ such that $I(u,v) \cap B = \emptyset$, and a bijection $g_N \in \mathcal{G}_n^v$ which agrees with g_M on the set B. Once again, by Lemma 3.4, the duplicator can turn his winning strategy in $G_{r''}(\mathfrak{C}_{n''})$ to a strategy for the continuation of $G_r(\mathfrak{C}_n)$, and he can use this strategy until the challenger removes some of the pebbles on elements of B. If this happens, then there are again at most $k-1$ pairs of pebbles on board after this R-move of the challenger, and so the duplicator can go on playing $G_r(\mathfrak{C}_n)$ without ever loosing. □

As the next step in proving the hierarchy theorem, we define for each n two non-isomorphic $\{V, E\}$-structures $\mathfrak{A}_{k,n}$ and $\mathfrak{B}_{k,n}$ by adding E-edges to $\mathfrak{C}_n$.

Definition 3.5 *For each positive integer n, let $\mathfrak{A}_{k,n} = (A_{k,n}, V^{\mathfrak{A}_{k,n}}, E^{\mathfrak{A}_{k,n}})$ and $\mathfrak{B}_{k,n} = (B_{k,n}, V^{\mathfrak{B}_{k,n}}, E^{\mathfrak{B}_{k,n}})$, where*

- $A_{k,n} = B_{k,n} = C_n$,
- $V^{\mathfrak{A}_{k,n}} = V^{\mathfrak{B}_{k,n}} = V^{\mathfrak{C}_n}$,
- $E^{\mathfrak{A}_{k,n}} = E^{\mathfrak{C}_n} \cup \{\overset{k}{a}\overset{k}{b} \in (\text{Row}_n)^k \times (\text{Row}_1)^k \cap (V^{\mathfrak{A}_{k,n}})^2 \mid \overset{k}{a} \in \text{Even} \iff \overset{k}{b} \in \text{Even}\}$, *and*

- $E^{\mathfrak{B}_{k,n}} = E^{\mathfrak{C}_n} \cup \{\overset{k}{a}\overset{k}{b} \in (\textsc{Row}_n)^k \times (\textsc{Row}_1)^k \cap (V^{\mathfrak{A}_{k,n}})^2 \mid \overset{k}{a} \in \textsc{Even} \iff \overset{k}{b} \in \textsc{Odd}\}$.

Thus, $(V^{\mathfrak{A}_{k,n}}, E^{\mathfrak{A}_{k,n}})$ is a directed graph isomorphic to the product of two disjoint cycles of length n with the complete graph K on 2^{k-1} vertices, whereas $(V^{\mathfrak{B}_{k,n}}, E^{\mathfrak{B}_{k,n}})$ is isomorphic to the product of one cycle of length $2n$ with K. In particular, $(V^{\mathfrak{B}_{k,n}}, E^{\mathfrak{B}_{k,n}})$ is connected, but $(V^{\mathfrak{A}_{k,n}}, E^{\mathfrak{A}_{k,n}})$ is non-connected:

Lemma 3.6 *Let φ_k be the sentence*

$$\forall\overset{k}{x}\forall\overset{k}{y}(V\overset{k}{x} \wedge V\overset{k}{y} \to [\mathrm{TC}_{\overset{k}{u}\overset{k}{v}} E\overset{k}{u}\overset{k}{v}]\overset{k}{x}\overset{k}{y})$$

of TC^k. Then $\mathfrak{B}_{k,n} \models \varphi_k$, but $\mathfrak{A}_{k,n} \not\models \varphi_k$.

On the other hand, for large enough n, the structures $\mathfrak{A}_{k,n}$ and $\mathfrak{B}_{k,n}$ cannot be distinguished by $\mathsf{PFP}^{k-1}(\mathbf{Q}_{2k-1})$-sentences of quantifier rank $\leq r$:

Proposition 3.7 *If $n = 2^r$, then the duplicator has a winning strategy in the game $AB_r^{k-1,2k-1}(\mathfrak{A}_{k,n}, \mathfrak{B}_{k,n})$.*

Proof. By Proposition 3.3, the duplicator has a winning strategy σ for the game $AB_r^{k-1,2k-1}(\mathfrak{C}_n, \mathfrak{C}_n)$ consisting of bijections from the set $\mathcal{G}_n$. Note that any $g_K \in \mathcal{G}_n$ is even at 1 and odd at n, and hence $\overset{k}{a}\overset{k}{b} \in E^{\mathfrak{A}_{k,n}} \iff g_K\overset{k}{a}\overset{k}{b} \in E^{\mathfrak{B}_{k,n}}$ whenever $\overset{k}{a} \in V^{\mathfrak{A}_{k,n}} \cap (\textsc{Row}_n)^k$ and $\overset{k}{b} \in V^{\mathfrak{A}_{k,n}} \cap (\textsc{Row}_1)^k$. But clearly this means that σ remains a winning strategy for the duplicator in $AB_r^{k-1,2k-1}$ even if the first copy of $\mathfrak{C}_n$ is replaced with $\mathfrak{A}_{k,n}$ and the second copy is replaced with $\mathfrak{B}_{k,n}$. $\square$

We conclude now that the TC^k-sentence φ_k of Lemma 3.6 is not expressible in $\mathsf{PFP}^{k-1}(\mathbf{Q}_{2k-1})$, and hence the proof of the Hierarchy Theorem 1.1 is complete.

Remark 3.8 *A different proof of this theorem, which uses essentially the methods introduced in [5] to construct structures playing the role of our $\mathfrak{C}_n$, can be found in [4].*

4 Deterministic transitive closure logic and its congruence closure

One reason for the interest in transitive closure logic is the fact (due to Immerman [9]) that on ordered finite structures a query is $\mathsf{NLogspace}$–computable if and only if it is definable in TC. Immerman also showed that there is a natural sublogic of TC that captures $\mathsf{Logspace}$ on ordered finite structures: *deterministic transitive closure logic* DTC.

For every formula $\varphi(\bar{x}, \bar{y})$ we consider its *deterministic version*

$$\varphi_D(\bar{x}, \bar{y}) := \varphi(\bar{x}, \bar{y}) \wedge \forall \bar{z}(\varphi(\bar{x}, \bar{z}) \rightarrow \bar{z} = \bar{y}).$$

DTC is the sublogic of TC whose formulae are those TC–formulae where the (TC)–rule is only applied to formulae of the form φ_D. We usually write $[\mathrm{DTC}_{\bar{x},\bar{y}}\varphi]\bar{u}, \bar{v}$ instead of $[\mathrm{TC}_{\bar{x},\bar{y}}\varphi_D]\bar{u}, \bar{v}$. Again we are interested in the *k–ary fragment* DTC^k which is the obvious restriction of TC^k to DTC.

Note that, similarly to the class $\mathcal{TC}$ (cf. Section 2.4), we can define a class $\mathcal{DTC}$ such that for each $k \geq 1$ we have $\mathsf{DTC}^k = \mathsf{FO}^k(Q_{\mathcal{DTC}})$.

We might ask if our hierarchy theorem extends to DTC, i.e. if for each $k \geq 1$ we have $\mathsf{DTC}^k \not\subseteq \mathsf{s\text{–}PFP}^{k-1}(\mathbf{Q}^{2k-1})$. It was shown in [5] that it does not, as a matter of fact we have $\mathsf{DTC} \subseteq \mathsf{s\text{–}PFP}^1$.

It has turned out that this is part of a general pattern: Whereas being closely related on ordered structures ($\mathsf{Logspace}$ and $\mathsf{NLogspace}$ cannot even be separated yet), DTC and TC behave quite differently on arbitrary structures (cf. [3]). One reason for this can be seen in the fact that DTC is not *congruence closed*. Taking up this observation, in the next subsection we introduce a suitable extension of DTC, its *vectorized congruence closure*, and show, in Subsection 4.3, that our hierarchy theorem extends to this logic.

4.1 Congruence closures

When talking of closure properties we have to make the notion of a logic precise: A *logic* is a regular logic in the sense of [1, 2] (i.e. it is closed under boolean combinations, particularization, and substitution of predicate symbols) which is, in addition, required to be finitary (i.e. each formula contains only finitely many distinct symbols). It is known that each logic can be represented in the form $\mathsf{FO}(\{Q_{\mathcal{C}} \mid \mathcal{C} \in \mathbf{C}\})$, where $\mathbf{C}$ is a class of isomorphism–closed classes of structures of a finite signature.

Given a structure $\mathfrak{A}$ and a congruence relation $\sim$ on $\mathfrak{A}$ we denote the factorization of $\mathfrak{A}$ through $\sim$ by $\mathfrak{A}/_{\sim}$. Following [10], we say that a logic L is *congruence closed* if for each class $\mathcal{C}$ definable in L the class

$$\mathcal{C}_* = \{(\mathfrak{A}, \sim) \mid \sim \text{ is a congruence relation on } \mathfrak{A}, \mathfrak{A}/_{\sim} \in \mathcal{C}\}$$

is also definable in L. The *congruence closure* L_* of L is the smallest congruence closed logic extending L.

Mekler and Shelah [10] observed that if $\mathsf{L} = \mathsf{FO}(\{Q_{\mathcal{C}} \mid \mathcal{C} \in \mathbf{C}\})$ then

$$\mathsf{L}_* = \mathsf{FO}(\{Q_{\mathcal{C}_*} \mid \mathcal{C} \in \mathbf{C}\}).$$

It can easily be seen that the logics TC, $\mathsf{s\text{–}PFP}$, and also their k–ary fragments TC^k, $\mathsf{s\text{–}PFP}^k$ are congruence closed. For let $\mathcal{C}$ be a class of σ–structures definable in any of these logics, say by a formula φ. Let $\sim$ be a relation symbol that is not contained in σ and let φ'_* be the formula obtained from φ by replacing each occurence of $=$ by $\sim$. Let γ be a formula saying that $\sim$ is a congruence relation and $\varphi_* = \varphi'_* \wedge \gamma$. Then a simple induction shows that φ_* defines $\mathcal{C}_*$.

On the other hand we will see that DTC is not congruence closed (cf. Corollary 4.3).

Another desirable property of a logic L is to be closed under vectorization, i.e. for each $k \geq 1$ and each class $\mathcal{C}$ definable in L, the class $\mathcal{C}^k$ is definable in L. In particular, when talking about arities it is reasonable to require a logic to have this property.

Note that DTC is closed under vectorization, whereas its congruence closure DTC_* is not (by Corollary 4.3).

So it seems natural to consider the *vectorized congruence closure* L^ω_* of a logic L, which is defined to be the smallest logic extending L which is closed under congruence and vectorization. For the reader interested in abstract model theory let us remark that the vectorized congruence closure of a logic is contained in its Δ–closure (cf. [1]) and hence preserves some nice properties of the logic such as compactness and Löwenheim–Skolem properties.

Unfortunately, there is no natural way to define the k–ary fragment of L^ω_* in a similar semantical manner. However, there is a natural definition of the k–ary fragment of L^ω_* if L is of the form $\mathsf{FO}(\{Q_\mathcal{C} \mid \mathcal{C} \in \mathbf{C}\})$ or $\mathsf{FO}^\omega(\{Q_\mathcal{C} \mid \mathcal{C} \in \mathbf{C}\})$. Observe first that in this case

$$\mathsf{L}^\omega_* = \mathsf{FO}^\omega(\{Q_{\mathcal{C}_*} \mid \mathcal{C} \in \mathbf{C}\}).$$

Now the k–ary fragment of L^ω_* is defined to be

$$\mathsf{L}^k_* = \mathsf{FO}^k(\{Q_{\mathcal{C}_*} \mid \mathcal{C} \in \mathbf{C}\}).$$

Observe that L^k_* is congruence closed. Hence $(\mathsf{L}^k)_* \subseteq \mathsf{L}^k_*$. By the same argument as in the proof of Corollary 4.3 it can be seen that in general the inclusion is strict (in particular for $\mathsf{L} = \mathsf{DTC}$).

Since every (regular, finitary) logic has a representation of the form $\mathsf{FO}^\omega(\{Q_\mathcal{C} \mid \mathcal{C} \in \mathbf{C}\})$ our definition covers all logics. However, in general the k–ary fragment depends on the representation. But for logics like $\mathsf{DTC} = \mathsf{FO}^\omega(Q_{\mathcal{DTC}})$ which are (essentially) given in the right representation the definition is completely natural. We have $\mathsf{DTC}^k_* = \mathsf{FO}^k(Q_{\mathcal{DTC}_*})$. Note that $\mathsf{DTC}^k_* \subseteq \mathsf{FO}(\mathbf{Q}^{2k})$.

4.2 The vectorized congruence closure and computational complexity

We have mentioned that the congruence closure of a logic often inherits nice model theoretic properties from the original logic. In this subsection we will see that this is also the case for "descriptive complexity theoretical" properties of the logic.

For the rest of this subsection we restrict ourselves to finite structures. Let C be one of the standard complexity classes $\mathsf{Logspace}$, $\mathsf{NLogspace}$, Ptime, NPtime, etc.

Since Turing machines need ordered structures as their input, we say that a class $\mathcal{C}$ of finite σ–structures is accepted in C iff the class

$$\mathcal{C}_< = \{(\mathfrak{A}, <) \mid \mathfrak{A} \in \mathcal{C}, < \text{ is a linear order of } \mathfrak{A}\}$$

is accepted in C. We say that a logic L is contained in C if each class of finite structures definable in L is accepted by a machine with resources in C.

Theorem 4.1 *Let L be a logic contained in C. Then L^ω_* is also contained in C.*

Proof. Given a Turing machine M with resources in C which accepts a class $\mathcal{C}$ definable in L, we first construct a machine M_* which accepts $\mathcal{C}_*$, using the following observation: If $\mathfrak{A}$ is a structure, $\sim$ is a congruence relation on $\mathfrak{A}$ and $<$ a linear order of its universe, then the substructure $\mathfrak{A}'$ generated by the $<$–smallest element of each $\sim$–class is isomorphic to $\mathfrak{A}/_\sim$. Note that $\mathfrak{A}'$ can be constructed from $\mathfrak{A}$ with resources in C.

It is now easy to pass from M_* to a machine M^k_* (still with resources in C) which accepts $\mathcal{C}^k_*$ (for any $k \geq 1$). □

Thus in particular DTC^ω_* still captures $\mathsf{Logspace}$ on ordered structures.

4.3 The extended hierarchy theorem

Theorem 4.2 *For each $k \geq 1$ we have $\mathsf{DTC}^k_* \not\subseteq \mathsf{s\text{–}PFP}^{k-1}(\mathbf{Q}^{2k-1})$.*

Proof. By the results of the Section 3 it suffices to show that for each $k \geq 1$ there is a sentence ψ_k of DTC^k_* such that $\mathfrak{A}_{k,n} \not\models \psi_k$ and $\mathfrak{B}_{k,n} \models \psi_k$ for each $n \geq 2$ (with $\mathfrak{A}_{k,n}$ and $\mathfrak{B}_{k,n}$ taken from Definition 3.5).

We first note that the $2k$-ary relation $\sim$ defined by

$$\rho(\overset{k}{x}, \overset{k}{y}) = (\neg V\overset{k}{x} \wedge \neg V\overset{k}{y}) \vee \exists \overset{k}{z}(E\overset{k}{x}\overset{k}{z} \wedge E\overset{k}{y}\overset{k}{z})$$

is an equivalence relation on $(A_{k,n})^k$ and $(B_{k,n})^k$ with $(2n+1)$ equivalence classes: One class is formed by all k–tuples not in $V^{\mathfrak{A}_{k,n}}$ ($V^{\mathfrak{B}_{k,n}}$ respectively), and for each $1 \leq i \leq n$ there is one class consisting of the even tuples in row i and one class consisting of the odd tuples in row i.

Actually, $\sim$ can be considered as a congruence relation on the structures $((A_{k,n})^k, V^{\mathfrak{A}_{k,n}}, E^{\mathfrak{A}_{k,n}})$ and $((B_{k,n})^k, V^{\mathfrak{B}_{k,n}}, E^{\mathfrak{B}_{k,n}})$ where we consider V as a unary relation and E as a binary one.

Letting $\mathfrak{A}_n$ ($\mathfrak{B}_n$) be the structure obtained from $\mathfrak{A}_{1,n}$ ($\mathfrak{B}_{1,n}$ respectively) by adding one new point not in V we have

$$((A_{k,n})^k, V^{\mathfrak{A}_{k,n}}, E^{\mathfrak{A}_{k,n}})/_\sim \cong \mathfrak{A}_n \quad \text{and} \quad ((B_{k,n})^k, V^{\mathfrak{B}_{k,n}}, E^{\mathfrak{B}_{k,n}})/_\sim \cong \mathfrak{B}_n.$$

Note that $\varphi = \forall u, v(Vu \wedge Vv \rightarrow [Q_{\mathcal{DTC}}xy, x, y \; Exy, x = u, y = v]$ is an $\mathsf{FO}(Q_{\mathcal{DTC}})$–sentence that holds in $\mathfrak{B}_n$ but not in $\mathfrak{A}_n$. Let $\mathcal{C}$ be the class of models of φ. Then we have

$$\mathfrak{A}_{k,n} \not\models [Q_{\mathcal{C}^k_*}\overset{k}{x}, \overset{k}{x}\overset{k}{y}, \overset{k}{x}\overset{k}{y} \; V\overset{k}{x}, E\overset{k}{x}\overset{k}{y}, \rho(\overset{k}{x}, \overset{k}{y})]$$

but

$$\mathfrak{B}_{k,n} \models [Q_{\mathcal{C}^k_*}\overset{k}{x}, \overset{k}{x}\overset{k}{y}, \overset{k}{x}\overset{k}{y} \; V\overset{k}{x}, E\overset{k}{x}\overset{k}{y}, \rho(\overset{k}{x}, \overset{k}{y})].$$

□

The following Corollary clarifies the relation between DTC, DTC_*, and DTC_*^ω:

Corollary 4.3 DTC_* *is not closed under vectorization. In particular this means that*

$$\mathsf{DTC} \subsetneq \mathsf{DTC}_* \subsetneq \mathsf{DTC}_*^\omega$$

Proof. As we mentioned earlier we have $\mathsf{DTC} \subseteq \mathsf{s\text{–}PFP}^1$. Hence $\mathsf{DTC}_* \subseteq (\mathsf{s\text{–}PFP}^1)_* = \mathsf{s\text{–}PFP}^1$. Now the claim follows from our extended hierarchy theorem. □

5 Further research

Let us conclude this paper by mentioning two open problems to which our hierarchy theorem gives solutions in some interesting special cases. Considering logics of the form $\mathsf{FO}^\omega(Q_\mathcal{C})$, the question is whether the hierarchy $(\mathsf{FO}^k(Q_\mathcal{C}))_{k\geq 1}$ (which we call the arity hierarchy of the quantifier $Q_\mathcal{C}$) is strict. $Q_{\mathcal{TC}}$ is an example of a quantifier whose arity hierarchy is strict, others can be found as easy consequences of our theorem:

- In [6] representations of inductive and partial fixed point logic of the form $\mathsf{FO}^\omega(Q_\mathcal{C})$ were given. The arity hierarchies of the corresponding quantifiers are strict by our theorem.
- The arity hierarchy of the Hamiltonian path quantifier $Q_{\mathcal{HP}}$ is strict. ($\mathcal{HP}$ is the class of $\{E, L, R\}$–structures that have a Hamiltonian path from the Left to the Right, i.e. a path in which every element of the structure occurs exactly once.)

  This is also an easy consequence of our theorem, since the structures $\mathfrak{B}_{k,n}$ essentially have a Hamiltonian E–path, whereas the structures $\mathfrak{A}_{k,n}$ have not.

- It was shown in [5] that the arity hierachy of DTC (hence of $Q_{\mathcal{DTC}}$) is strict.

On the other hand there are well–known quantifiers whose arity hierarchy collapses. For example, consider the class

$$\mathcal{EVEN} = \{(A, P^{\mathfrak{A}}) \mid P^{\mathfrak{A}} \text{ is a finite set of even cardinality.}\}$$

It is easy to see that $\mathsf{FO}^{\omega}(Q_{\mathcal{EVEN}}) = \mathsf{FO}(Q_{\mathcal{EVEN}})$ (cf. [11]). In fact it is not hard to give examples of quantifiers whose arity hierarchy collapses on an arbitrary level (cf. [4] for more details).

So we have the following

Open Problem 5.1 *Find criteria for the arity hierarchy of a Lindström quantifier to be strict.*

An even stronger notion (introduced in [7]) than the arity hierarchy of a quantifier $Q_{\mathcal{C}}$ being strict is that $\mathsf{FO}^{\omega}(Q_{\mathcal{C}}) \not\subseteq \mathsf{FO}(\mathbf{Q}^{l})$ for any $l \geq 1$. Consequently, in this case we say that the arity hierarchy of $Q_{\mathcal{C}}$ is *strong*.

Note that we actually proved that the arity hierarchies of $Q_{\mathcal{TC}}$ and $Q_{\mathcal{DTC}_*}$ are strong.

Again we have:

Open Problem 5.2 *Find criteria for the arity hierarchy of a Lindström quantifier to be strong.*

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