Bounded fixed-parameter tractability and $\log^2 n$ nondeterministic bits

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Abstract

Motivated by recent results showing that there are natural parameterized problems that are fixed-parameter tractable, but can only be solved by fixed-parameter tractable algorithms the running time of which depends non-elementarily on the parameter, we propose a notion of *bounded fixed-parameter tractability*, where the dependence of the running time on the parameter is restricted to be singly exponential.

We develop a basic theory that is centred around the class EPT of tractable problems and an EW-hierarchy of classes of intractable problems, both in the bounded sense. By and large, this theory is similar to the established *unbounded* parameterized complexity theory, but there are some remarkable differences. Most notably, certain natural model-checking problems that are known to be fixed-parameter tractable in the unbounded sense have a very high complexity in the bounded theory. The problem of computing the VC-dimension of a family of sets, which is known to be complete for the class W$[1]$ in the unbounded theory, is complete for the class EW$[3]$ in the bounded theory.

It turns out that our bounded parameterized complexity theory is closely related to the classical complexity theory of problems that can be solved by a nondeterministic polynomial time algorithm that only uses $\log^2 n$ nondeterministic bits, and in particular to the classes LOGSNP and LOGNP introduced by Papadimitriou and Yannakakis.

1. Introduction

The idea of fixed-parameter tractability is to approach hard algorithmic problems by isolating problem parameters that can be expected to be small in certain applications and then develop algorithms that are polynomial except for an arbitrary dependence on the parameter. More precisely, a problem is fixed-parameter tractable if it can be solved by an algorithm the running time of which is bounded by $f(k) \cdot p(n)$, where n denotes the size of the input, k the parameter, f is an arbitrary computable function, and p a polynomial. Since the choice of suitable parameters allows for a great

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flexibility, fixed-parameter algorithms have found their way into practical applications such diverse as computational biology, database systems, computational linguistics, and automated verification (cf. [3]). On the theoretical side, a theory of parameterized intractability has been developed that led to a comprehensive classification of parameterized problems into tractable and hard problems (cf. [6, 3]).

Allowing an arbitrary computable function f in the running time bound of a fixed-parameter tractable algorithm seems questionable, though. A running time of $2^{2^k}n$ cannot really be considered "tractable" even for small values of k (say, $k \leq 10$). The standard and to some extent valid response to such objections is that (a) for natural problems, such extreme parameter dependence rarely occurs and (b) to obtain a robust theory, one has to compromise. Referring to the "classical" class of tractable problems, polynomial time, one may add that (c) an algorithm with a running time of $O(n^{100})$ cannot be considered "tractable" either, even though it is a polynomial time algorithm. However, recent results due to Frick and the second author [12] show that the crucial point (a) has important exceptions: There are natural fixed-parameter tractable problems that cannot be solved by an algorithm whose running time is bounded by $f(k) \cdot \text{poly}(n)$ for any elementary function f. These problems are so-called model-checking problems; database query evaluation is an application that can be described by such problems [13]. The results imply that the running time of the fixed-parameter tractable algorithm obtained from Courcelle's well-known theorem [2] that monadic second-order properties of graphs of bounded tree-width can be decided in linear time also has a non-elementary dependence on the parameter. Courcelle's theorem has been viewed a centrepiece of parameterized complexity theory (a long chapter in Downey and Fellows' monograph [6] is devoted to Courcelle's theorem). This raises some doubts about parts of the theory of fixed-parameter tractability. Of course, intractability results with respect to this liberal definition are stronger. Moreover these doubts by no means diminish the value of the practical work on fixed-parameter tractable algorithms; algorithms developed in this context often have running times $c^k \cdot n$ for some constant c with $1 < c \leq 2$.

The important fact is that there are viable alternatives to the notion of fixed-parameter tractability: One can simply put upper bounds on the growth of the "parameter dependence" f, the two most natural being $f \in 2^{\text{poly}(k)}$ and the stricter $f \in 2^{O(k)}$. The resulting *bounded fixed-parameter tractability* classes are still fairly robust, and they contain all of the problems that are "fixed-parameter tractable in practice". While we do not want to propose an industry generating papers on various bounded parameterized complexity theories, we hope that our results will convince the reader that at least the bounded theory we consider here is well worth being explored.

We study the stricter notion of bounded fixed-parameter tractability. We let EPT be the class of all parameterized problems that can be solved in time $2^{O(k)} \cdot \text{poly}(n)$. We introduce a suitable notion of *ept-reduction* and define the class EW[P] and a hierarchy of classes EW$[t]$, for $t \geq 1$, within EW[P] corresponding to the class W[P] and to the classes of the W-hierarchy of *unbounded* parameterized complexity.[1] We observe

[1]Some remarks on our terminology may be helpful here: *Classical complexity theory* refers to the standard, unparameterized, complexity theory. In parameterized complexity, we distinguish between the usual theory, referred to as *unbounded parameterized complexity theory*, and the *bounded parameterized complexity theory* developed here.

that, for all $t \geq 1$, if W$[t] \neq$ FPT then EW$[t] \neq$ EPT. So we can assume that the EW-hierarchy does not collapse to EPT (that is, if we believe the assumption of the unbounded theory that the W-hierarchy does not collapse to FPT). We prove that the logical characterisations of the W-hierarchy [7, 10, 11] can be transfered to the bounded EW-hierarchy, which shows that the classes have a certain robustness. It has to be said, though, that the EW-hierarchy is less robust than the W-hierarchy. This is particularly true for the first level EW$[1]$ of the hierarchy.

We then consider a few complete problems for our classes. Many completeness results can easily be transfered from the unbounded to the bounded theory. As an example, we prove that the parameterized dominating set problem, which is W$[2]$-complete under fpt-reductions, is EW$[2]$-complete under ept-reductions. A surprise occurs when we consider a parameterized version of the problem of computing the VC-dimension of a family of sets. In the unbounded theory, this problem is known to be W$[1]$-complete under fpt-reductions. We prove that in our bounded theory, VC-dimension is EW$[3]$-complete under ept-reductions. Thus we are in the odd situation that in the unbounded theory, VC-dimension is "easier" than dominating set, whereas in the bounded theory, it is "harder". The completeness of the parameterized VC-dimension problem for the third level of our hierarchy seems very natural in view of Schaefer's result that a classical version of the VC-dimension problem, where the family of sets is represented succinctly, is complete for the third level of the polynomial hierarchy [18].

Less surprisingly, we prove that the (unbounded) fixed-parameter tractable model-checking problems that have been shown to have no fixed-parameter tractable algorithms with elementary parameter dependence in [12] are complete for natural intractable classes in the bounded theory. Specifically, we prove that model-checking for first-order logic on words is complete for the class EAW$[*]$, the bounded analogue of the class AW$[*]$.

One of the nicest features of our bounded theory is that it is intimately linked to the classical complexity class NP[$\log^2 n$] of all problems that can be solved by a nondeterministic polynomial time algorithm that uses only $O(\log^2 n)$ nondeterministic bits. There are several natural examples of such problems. The best known may be the problem of computing the VC-dimension of a given family of sets [15] and the hypergraph traversal problem [8]. Papadimitriou and Yannakakis [15] introduced two syntactically defined complexity classes LOGSNP and LOGNP and proved that many natural problems are complete for one of these classes. The definition of these classes is reminiscent of some of the logical characterisations of the classes of the W-hierarchy and the EW-hierarchy. Motivated by this observation, we introduce a hierarchy of classical complexity classes $\text{LOG}[t]$, for $t \geq 2$, which may be viewed as restrictions of the corresponding classes EW$[t]$ to the parameter value $\log\ n$, where n denotes the size of the input. We prove that LOGSNP $= \text{LOG}[2]$ and LOGNP $= \text{LOG}[3]$. Thus our classes put Papadimitriou and Yannakakis's classes into a larger context. We show that NP[$\log^2 n$] = PTIME if and only if EW[P] = EPT and that for all $t \geq 2$ we have $\text{LOG}[t] =$ PTIME if and only if EW$[t] =$ EPT. This establishes a nice direct connec-

Furthermore, we distinguish between *classical problems*, which are just languages $Q \subseteq \Sigma^*$ over some finite alphabet Σ, and *parameterized problems*, which are pairs (Q, κ), where $Q \subseteq \Sigma^*$ is a classical problem and $\kappa : \Sigma^* \to \mathbb{N}$ a *parameterization*.

tion between classical complexity theory and our bounded parameterized theory; no such connection is known for the W-hierarchy (and it probably does not exist).

Our paper is organised as follows: In Section 2 we review the basic notions of (unbounded) parameterized complexity theory and at the same time introduce the corresponding notions of the bounded theory. In Section 3, we relate the class EW[P] with limited nondeterminism. In Section 4, we give logical characterisations of the EW-hierarchy, and in Section 5 we prove two basic completeness results for the classes EW$[2]$ and EW$[3]$. Section 6 is devoted to the connection between the classes of the EW-hierarchy and the classical classes of problems that can be solved with $\log^2 n$ nondeterministic bits introduced by Papadimitriou and Yannakakis [15]. In Section 7, we study higher levels of intractability in our bounded theory. Finally, Sections 8 and 9 are devoted to the part of the theory that is not so nice. We introduce a matrix of classes EW$[t, d]$ generalising the EW-hierarchy and identify a class within this matrix that seems a good candidate for a class EW$[1]$. We prove that the parameterized clique problem is complete for this class.

2. The basic notions

2.1. FPT and EPT. Let Σ be a finite alphabet. A *parameterized problem* (over the alphabet Σ) is a pair (Q, κ) consisting of a set $Q \subseteq \Sigma^*$ of strings over Σ and a polynomial time computable function $\kappa : \Sigma^* \to \mathbb{N}$, the *parameterization*. Any $x \in \Sigma^*$ is called an *instance* of Q and $\kappa(x)$ is the corresponding *parameter*.

Hence, a parameterized problem consists of a problem in the usual complexity theoretic sense together with a parameterization.

For example, choose a finite alphabet Σ such that propositional formulas are strings over Σ in a natural way. The parameterized problem p-SAT is the problem (Q, κ), where Q is the set of satisfiable propositional formulas and $\kappa : \Sigma^* \to \mathbb{N}$ is defined by

$$\kappa(x) := \begin{cases} \text{number of variables of } x, & \text{if } x \text{ is a propositional formula} \\ 0, & \text{otherwise.} \end{cases}$$

The following notation for p-SAT illustrates how we normally present parameterized problems:

p-SAT	
Instance:	A propositional formula α.
Parameter:	The number of variables of α.
Problem:	Decide whether α is satisfiable.

Definition 1. Let $\mathfrak{F}$ be a set of total functions from $\mathbb{N}$ to $\mathbb{N}$. A parameterized problem (Q, κ) over the alphabet Σ is $\mathfrak{F}$*-fixed-parameter tractable*, if there is a function $f \in \mathfrak{F}$, a polynomial $p \in \mathbb{N}[X]$, and an algorithm that, given $x \in \Sigma^*$, decides whether $x \in Q$ in at most $f(\kappa(x)) \cdot p(|x|)$ steps.

We denote the class of all $\mathfrak{F}$-fixed-parameter tractable problems by $\mathfrak{F}$-FPT.

The standard notion of fixed-parameter tractability is based on the class $\mathfrak{R}$ of all computable functions. We use the standard terminology and denote the class $\mathfrak{R}$-FPT simply by FPT.[2] We usually refer to the "standard" parameterized complexity theory based on the class FPT as *unbounded (parameterized complexity) theory*, to distinguish it from *bounded theories* based on $\mathfrak{F}$ for "bounded" classes $\mathfrak{F}$, $\mathfrak{F} \subset \mathfrak{R}$.

In this paper, we are mainly interested in $\mathfrak{E}$-fixed-parameter tractability, where $\mathfrak{E}$ is the set of computable functions in $2^{O(k)}$. To simplify the notation, we write EPT instead of $\mathfrak{E}$-FPT. Further natural and interesting classes are $\mathfrak{SUBE}$-FPT and $\mathfrak{EXP}$-FPT, where $\mathfrak{SUBE} = 2^{o(k)}$ (more precisely, $\mathfrak{SUBE}$ is the class of all computable functions in $2^{o(k)}$) and $\mathfrak{EXP} = 2^{\text{poly}(k)}$. The latter has been investigated in [19]. If $\mathfrak{F} = O(1)$ is the set of all constant functions, then $\mathfrak{F}$-FPT is PTIME, or more precisely, $\mathfrak{F}$-FPT is the class of parameterized problems (Q, κ) with Q in PTIME.

Clearly, if $\mathfrak{F} \subseteq \mathfrak{F}'$ then $\mathfrak{F}$-FPT $\subseteq \mathfrak{F}'$-FPT and hence, every problem in EPT is in FPT. An example of a problem in EPT is p-SAT, where we can choose as f the function $f(k) := 2^k$. If $Q \subseteq \Sigma^*$ is a decidable problem that is not decidable in time $2^{O(n)}$ and $\kappa : \Sigma^* \to \mathbb{N}$ is defined by $\kappa(x) = |x|$, then the parameterized problem (Q, κ) is in FPT$\setminus$EPT. *Natural* problems in FPT$\setminus$EPT are known to exist under certain complexity theoretic assumptions:

$$\begin{aligned} p\text{-MC}(\text{WORDS}, \text{FO}) &\in \text{FPT} \setminus \text{EPT} \quad \text{if FPT} \neq \text{AW}[*], \\ p\text{-MC}(\text{WORDS}, \text{MSO}) &\in \text{FPT} \setminus \text{EPT} \quad \text{if P} \neq \text{NP} \end{aligned}$$

(cf. [12]). Here, p-MC(WORDS, FO) and p-MC(WORDS, MSO) denote the parameterized model-checking problem for the class of words and first-order logic FO and the class of words and monadic second-order logic MSO, respectively (compare 4.3 for the definition of model-checking problems).

2.2. Reductions. To compare the complexities of parameterized problems that are not $\mathfrak{F}$-fixed-parameter tractable, we need a notion of reduction. We only consider many-one reductions. The crucial property expected from a notion of reduction for $\mathfrak{F}$-FPT is:

$$\text{If } (Q, \kappa) \text{ is reducible to } (Q', \kappa') \text{ and } (Q', \kappa') \in \mathfrak{F}\text{-FPT, then } (Q, \kappa) \in \mathfrak{F}\text{-FPT.} \tag{1}$$

We give the definitions for the cases we are interested in here, FPT and EPT.

Definition 2. Let (Q, κ) and (Q', κ') be parameterized problems over the alphabets Σ and Σ', respectively. A *reduction* from (Q, κ) to (Q', κ') is a function $R : \Sigma^* \to (\Sigma')^*$ with

$$Qx \iff Q'\, R(x)$$

for all $x \in \Sigma^*$.

1. R is an *fpt-reduction* if there are computable functions f, g and a polynomial p such that
   (a) $R(x)$ is computable in time $f(\kappa(x)) \cdot p(|x|)$,
   (b) $\kappa'(R(x)) \leq g(\kappa(x))$ for all $x \in \Sigma^*$.

[2] Sometimes, FPT is even defined as $\mathfrak{A}$-FPT, where $\mathfrak{A}$ denotes the class of all functions $f : \mathbb{N} \to \mathbb{N}$. (Downey and Fellows [6] call $\mathfrak{R}$-FPT *strongly uniform FPT* and $\mathfrak{A}$-FPT *uniform-FPT*.) However, $\mathfrak{R}$-FPT is a more robust class.

2. R is an *ept-reduction* if there are constants $c, d \geq 0$ and a polynomial p such that
   (a) $R(x)$ is computable in time $2^{c \cdot \kappa(x)} \cdot p(|x|)$,
   (b) $\kappa'(R(x)) \leq d \cdot (\kappa(x) + \log |x|)$ for all $x \in \Sigma^*$.

It is easy to verify (1) for fpt-reducibility with respect to FPT and for ept-reducibility with respect to EPT.

Many reductions presented in this paper are fpt-reductions *and* ept-reductions, in fact they are efpt-reductions in the sense of the following definition.

Definition 3. A reduction R from (Q, κ) to (Q', κ') is an *efpt-reduction*, if there are constants $c, d \geq 0$ and a polynomial p such that

1. $R(x)$ is computable in time $2^{c \cdot \kappa(x)} \cdot p(|x|)$,
2. $\kappa'(R(x)) \leq d \cdot \kappa(x)$ for all $x \in \Sigma^*$.

We write $(Q, \kappa) \leq^{\text{ept}} (Q', \kappa')$ if there is an ept-reduction from (Q, κ) to (Q', κ') and $(Q, \kappa) \equiv^{\text{ept}} (Q', \kappa')$ if $(Q, \kappa) \leq^{\text{ept}} (Q', \kappa')$ and $(Q', \kappa') \leq^{\text{ept}} (Q, \kappa)$. We let

$$[(Q, \kappa)]^{\text{ept}} = \{(Q', \kappa') \mid (Q', \kappa') \leq^{\text{ept}} (Q, \kappa)\}.$$

Analogously, we define $\leq^{\text{fpt}}, \equiv^{\text{fpt}}, [(Q, \kappa)]^{\text{fpt}}, \leq^{\text{efpt}}$ and $\equiv^{\text{efpt}}$.

The notions of fpt-reduction and ept-reduction are incomparable: To see this, let $Q \subseteq \Sigma^*$ be a problem that is not in polynomial time. Let $\kappa, \kappa' : Q \to \mathbb{N}$ be defined by $\kappa(x) = 1$ and $\kappa'(x) = \log |x|$ for all $x \in \Sigma^*$. Then clearly (Q, κ) is ept-reducible to (Q, κ'). However, it is easy to see that if (Q, κ) were fpt-reducible to (Q, κ') then Q would be in polynomial time.

Conversely, let (Q, κ) be any problem in FPT $\setminus$ EPT, and let (Q', κ') be any non-trivial problem in EPT (non-trivial meaning that Q' is neither the empty set nor the set of all strings over a given alphabet). Then (Q, κ) is fpt-reducible, but not ept-reducible to (Q', κ').

In Section 7, we will see natural problems (Q, κ) and (Q', κ') such that

$$(Q, \kappa) \in \text{FPT}, \;\; (Q', \kappa') \leq^{\text{ept}} (Q, \kappa), \;\; \text{and} \;\; (Q', \kappa') \notin \text{FPT}.$$

The incomparibility of ept- and fpt-reducibility is the source of the richnes of the EPT-theory in much the same way as the incomparibility of fpt-reductions and polynomial reductions is for the FPT-theory. The introduction of efpt-reductions is of pragmatic nature: As already mentioned, it just happens that many reductions we consider are efpt-reductions.

3. The class EW[P] and limited nondeterminism

In this section we relate the class EW[P], the bounded analogue of the class W[P], with limited nondeterminism.

First, we recall the definition of W[P] (cf. [1, 6]). For this purpose, we consider *circuits*. They are defined in the standard way. To be a bit more specific, let us say that our circuits consist of *input gates*, *and gates*, and *or gates* of arbitrary finite arity, and *not gates* and they have one designated output node (that is, they only compute

Boolean functions). The *weight* of a truth value assignment to the input nodes of C is the number of input nodes set to TRUE. By definition,

$$\mathrm{W[P]} = [p\text{-}\mathrm{CIRC}]^{\mathrm{fpt}},$$

where

p-CIRC	
Instance:	A circuit C and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Decide whether C has a satisfying assignment of weight k.

By analogy, we define EW[P] by

$$\mathrm{EW[P]} := [p\text{-}\mathrm{CIRC}]^{\mathrm{ept}}.$$

By $\mathrm{NP}[\log^2 n]$ we denote the classical complexity class of all problems that can be solved by nondeterministic polynomial time algorithms using only $O(\log^2 n)$ nondeterministic bits (in the Kintala-Fischer model of limited nondeterminism [14]). The result relating EW[P] with limited nondeterminism reads as follows:

Theorem 4. $\mathrm{EW[P]} = \mathrm{EPT}$ *if and only if* $\mathrm{NP}[\log^2 n] = \mathrm{PTIME}$.

Proof: First assume that $\mathrm{EW[P]} = \mathrm{EPT}$. Let Q be a (classical) problem decided by a nondeterministic polynomial time machine $\mathbb{M}$ that on every run on any input x performs at most $d \cdot \log^2 |x|$ nondeterministic steps (for some constant $d \in \mathbb{N}$). We may suppose that $\mathbb{M}$ first carries out the nondeterministic steps and that they altogether consist in appending to the input x a 0–1 string of length $d \cdot \log^2 |x|$. (If we write $\log n$ where an integer is expected, we mean $\lceil \log n \rceil$.)

The deterministic part of the computation of $\mathbb{M}$ on input x can be simulated by a circuit C_x in the standard way (e.g., compare the proof of Theorem 8.1 in [16]) such that

$$\mathbb{M} \text{ accepts } x \iff C_x \text{ has a satisfying assignment.} \tag{2}$$

The size of C_x is polynomial in $|x|$ and C_x has $d \cdot \log^2 |x|$ input nodes corresponding to the 0–1 string chosen in the nondeterministic part of the computation of $\mathbb{M}$.

Now we apply what Downey and Fellows in [6] call the $k \cdot \log n$ trick: We think of the $d \cdot \log^2 |x|$ input nodes of C_x as being arranged in $d \cdot \log |x|$ blocks of $\log |x|$ nodes. We construct the circuit D_x by adding $d \cdot \log |x|$ blocks, each of $|x|$ new input nodes, to C_x and by ensuring (with additional gates) that at most one input node of each new block can be set to TRUE (in a satisfying assignment of D_x). Moreover, we wire the new input nodes with the old input nodes (i.e., the input nodes of C_x) in such a way that the following holds: If the jth input node of the ith block of D_x is set to TRUE then exactly those old input nodes of the ith block, which correspond to positions of the binary representation of j carrying a 1, are set to TRUE. Then

$$\begin{aligned} & C_x \text{ has a satisfying assignment} \\ \iff \; & D_x \text{ has a satisfying assignment of weight } d \cdot \log |x| \end{aligned}$$

and therefore,

$$Qx \quad \Longleftrightarrow \quad (D_x, d \cdot \log|x|) \in p\text{-}\textsc{Circ}.$$

By our assumption EW[P] $=$ EPT, we can decide whether $(D_x, d \cdot \log|x|) \in p\text{-}\textsc{Circ}$ in time $2^{c \cdot d \cdot \log|x|} \cdot p(|D_x|)$ (for some constant c and polynomial p) and hence, in time polynomial in $|x|$.

For the converse direction assume that NP$[\log^2 n] =$ PTIME. We show that $p\text{-}\textsc{Circ} \in$ EPT. Let (C, k) be an instance of $p\text{-}\textsc{Circ}$ of size n. Note that it can be decided whether C is k-satisfiable by a nondeterministic polynomial time algorithm that uses $k \cdot \log n$ nondeterministic bits: First, it guesses k input nodes, which requires $\log n$ nondeterministic bits for each input node, and then it deterministically checks if the corresponding assignment satisfies C.

Thus the restriction of $p\text{-}\textsc{Circ}$ to input instances (C, k) with $k \leq \log n$, where n is the size of C, is in NP$[\log^2 n] =$ PTIME and thus in EPT.

For instances (C, k) with $k > \log n$, let C' be the circuit obtained from C by adding a new output node, which is an or-node of fan in 2^k, each input line coming from the output node of a copy of C. All these copies only share the input nodes. The circuit C' can be obtained in time $O(2^k \cdot |C|)$ and the size n' of the instance (C', k) is at least 2^k. Moreover,

$$\begin{aligned} & C' \text{ has a satisfying assignment of weight } k \\ \Longleftrightarrow \quad & C \text{ has a satisfying assignment of weight } k. \end{aligned}$$

This reduces the general problem to the problem for instances with $k \leq \log n$ □

4. Logical characterisations of the EW-hierarchy

In this section, after introducing the classes of the EW-hierarchy, we present characterisations of these classes, first in terms of variants of the weighted satisfiability problems defining the classes, then in terms of model-checking problems for first-order logic and finally, in terms of Fagin-definable problems. Most results (and their proofs) are extensions or refinements of the corresponding characterisations of the W-hierarchy.

4.1. The W-hierarchy and the EW-hierarchy. In unbounded parameterized complexity theory, the classes of the W-hierarchy were originally defined by means of weighted satisfiability problems for propositional logic. We recall the definition and extend it to EPT.

Formulas of propositional logic are built up from *propositional variables* $X_1, X_2, \ldots$ by taking conjunctions, disjunctions, and negations. The negation of a formula α is denoted by $\neg\alpha$. We distinguish between *small conjunctions*, denoted by $\wedge$, which are just conjunctions of two formulas, and *big conjunctions*, denoted by $\bigwedge$, which are conjunctions of arbitrary finite sequences of formulas. Analogously, we distinguish between *small disjunctions*, denoted by $\vee$, and *big disjunctions*, denoted by $\bigvee$. Every formula has a naturally defined *syntax tree*, and the size $|\alpha|$ of a formula α is the number of nodes of the syntax tree of α.

The *weight* of an assignment is the number of variables set to TRUE. A propositional formula α is k*-satisfiable* (where $k \in \mathbb{N}$), if there is an assignment for the set of variables of α of weight k satisfying α.

For a set Γ of propositional formulas, the *parameterized weighted satisfiability problem* WSAT(Γ) *for formulas in* Γ is the following parameterized problem:

p-WSAT(Γ)	
Instance:	A propositional formula $\alpha \in \Gamma$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Decide whether α is k-satisfiable.

For $t \geq 0$ and $d \geq 1$ define the sets $\Gamma_{t,d}$ and $\Delta_{t,d}$ by induction on t (here, by $(\lambda_1 \wedge \ldots \wedge \lambda_r)$ we mean the iterated small conjunction $((\ldots(\lambda_1 \wedge \lambda_2) \wedge \ldots) \wedge \lambda_r)$:

$$\begin{aligned}
\Gamma_{0,d} &:= \{(\lambda_1 \wedge \ldots \wedge \lambda_r) \mid \lambda_1, \ldots, \lambda_r \text{ literals and } r \leq d\}, \\
\Delta_{0,d} &:= \{(\lambda_1 \vee \ldots \vee \lambda_r) \mid \lambda_1, \ldots, \lambda_r \text{ literals and } r \leq d\}, \\
\Gamma_{t+1,d} &:= \{\textstyle\bigwedge_{i \in I} \delta_i \mid I \text{ a finite set and } \delta_i \in \Delta_{t,d} \text{ for all } i \in I\}, \\
\Delta_{t+1,d} &:= \{\textstyle\bigvee_{i \in I} \gamma_i \mid I \text{ a finite set and } \gamma_i \in \Gamma_{t,d} \text{ for all } i \in I\}.
\end{aligned}$$

If in the definition of $\Gamma_{0,d}$ and $\Delta_{0,d}$ we require that all literals are positive (negative) we obtain the sets denoted by $\Gamma^+_{t,d}$ and $\Delta^+_{t,d}$ ($\Gamma^-_{t,d}$ and $\Delta^-_{t,d}$), respectively.

In unbounded parameterized complexity the classes $\mathrm{W}[1], \mathrm{W}[2], \ldots$ constitute the W-hierarchy; for $t \geq 2$,

$$\mathrm{W}[t] = [p\text{-WSAT}(\Gamma_{t,1})]^{\mathrm{fpt}},$$

so we define:

$$\mathrm{EW}[t] := [p\text{-WSAT}(\Gamma_{t,1})]^{\mathrm{ept}}.$$

It was realized that $\mathrm{W}[1]$ can be conveniently defined by

$$\mathrm{W}[1] = [p\text{-WSAT}(\Gamma_{1,2})]^{\mathrm{fpt}},$$

so we define:

$$\mathrm{EW}[1] := [p\text{-WSAT}(\Gamma_{1,2})]^{\mathrm{ept}}.$$

The classes $\mathrm{EW}[1], \mathrm{EW}[2], \ldots$ constitute the EW-hierarchy of $\mathfrak{E}$-parameterized complexity theory.

The classes $\mathrm{W}[t]$ are robust in the following sense: For all $t, d \geq 1$ with $t + d \geq 3$,

$$\mathrm{W}[t] = [p\text{-WSAT}(\Gamma_{t,d})]^{\mathrm{fpt}}.$$

This robustness does not seem to be shared by the classes $\mathrm{EW}[t]$. Instead, we have to consider a matrix EW of classes given by

$$\mathrm{EW}[t, d] := [p\text{-WSAT}(\Gamma_{t,d})]^{\mathrm{ept}}.$$

In this paper we will mostly deal with the case $d = 1$, only Section 8 is devoted to the full EW-matrix. In this section we mainly consider the case $t \geq 2$; the class $\mathrm{EW}[1]$ will be analysed in Section 9.

4.2. Propositional logic. The notions of complete and hard problem for a complexity class are defined in the usual fashion; they refer to fpt-reductions or to ept-reductions depending on whether we consider a class of unbounded parameterized complexity theory or a class of $\mathfrak{E}$-parameterized complexity theory.

For even (odd) t already the weighted satisfiability problem for monotone (anti-monotone) propositional formulas is complete for $\mathrm{EW}[t]$:

Theorem 5. *1.* p-WSAT$(\Gamma^+_{t,1})$ *is complete for* $\mathrm{EW}[t]$ *for even* $t > 1$.

2. p-WSAT$(\Gamma^-_{t,1})$ *is complete for* $\mathrm{EW}[t]$ *for odd* $t > 1$.

For further reference we prove a stronger version of this theorem. We start with some technical remarks. Often, we will tacitly make use of the following fact:

Lemma 6. *Let* $t, d \geq 1$. *For even* t *there is a polynomial time algorithm associating with every* $\alpha \in \Gamma_{t,d}$ *an equivalent formula in* $\Gamma_{t,d}$ *of the form*

$$\bigwedge_{i_1 \in I} \bigvee_{i_2 \in I} \cdots \bigwedge_{i_{t-1} \in I} \bigvee_{i_t \in I} \beta_{i_1,\dots,i_t}.$$

where $\beta_{i_1,\dots,i_t}$ *is in* $\Gamma_{0,d}$. *The corresponding result holds for odd* t.

A useful tool in some proofs will be a variant of the weighted satisfiability problem, namely the *parameterized partitioned satisfiability problem* p-PSAT(Γ); here, Γ is a class of propositional formulas and

p-PSAT(Γ)	
Instance:	A formula $\alpha \in \Gamma$ and a partition $(\mathcal{X}_m)_{1 \leq m \leq k}$ of the variables of α.
Parameter:	k (the number of sets in the partition).
Problem:	Decide whether $(\alpha, (\mathcal{X}_m)_{1 \leq m \leq k})$ is *satisfiable*, that is, whether α has a satisfying assignment that sets exactly one variable of each $\mathcal{X}_m$ to TRUE.

By the next lemmas we show, for $t, d \geq 1$,

$$p\text{-WSAT}(\Gamma_{t,d}) \leq^{\text{efpt}} p\text{-PSAT}(\Gamma^+_{t,d}) \leq^{\text{efpt}} p\text{-WSAT}(\Gamma^+_{t,d}) \leq^{\text{efpt}} p\text{-WSAT}(\Gamma_{t,d}) \quad \text{for even } t, \tag{3}$$

$$p\text{-WSAT}(\Gamma_{t,d}) \leq^{\text{efpt}} p\text{-PSAT}(\Gamma^-_{t,d}) \leq^{\text{efpt}} p\text{-WSAT}(\Gamma^-_{t,d}) \leq^{\text{efpt}} p\text{-WSAT}(\Gamma_{t,d}) \quad \text{for odd } t, \tag{4}$$

which together imply the theorem.

The statements

$$p\text{-WSAT}(\Gamma^+_{t,d}) \leq^{\text{efpt}} p\text{-WSAT}(\Gamma_{t,d})$$

and

$$p\text{-WSAT}(\Gamma^-_{t,d}) \leq^{\text{efpt}} p\text{-WSAT}(\Gamma_{t,d})$$

are trivial. The remaining statements follow from the following two lemmas.

Lemma 7. *For* $t, d \geq 1$*:*

1. p-PSAT$(\Gamma^+_{t,d}) \leq^{\text{efpt}} p$-WSAT$(\Gamma^+_{t,d})$ *for even* t.
2. p-PSAT$(\Gamma^-_{t,d}) \leq^{\text{efpt}} p$-WSAT$(\Gamma^-_{t,d})$ *for odd* t.

Proof: Since p-PSAT$(\Gamma^-_{1,1})$ is in EPT, the case $t = d = 1$ is clear. So let us assume that $t + d \geq 3$.

Let $(\alpha, (\mathcal{X}_m)_{1 \leq m \leq k})$ be an instance of the corresponding partitioned satisfiability problem. The fact that an assignment of weight k sets exactly one variable from each $\mathcal{X}_m$ to TRUE can be expressed by

$$\alpha^+ = \bigwedge_{1 \leq m \leq k} \bigvee_{X \in \mathcal{X}_m} X \quad \text{and by} \quad \alpha^- = \bigwedge_{1 \leq m \leq k} \bigwedge_{\substack{X, X' \in \mathcal{X}_m \\ X \neq X'}} (\neg X \vee \neg X').$$

If $\alpha \in \Gamma^+_{t,d}$ then $(\alpha \wedge \alpha^+)$ is equivalent to a formula $\beta^+ \in \Gamma^+_{t,d}$ and, if $\alpha \in \Gamma^-_{t,d}$ then $(\alpha \wedge \alpha^-)$ is equivalent to a formula $\beta^- \in \Gamma^-_{t,d}$ (if $d = 1$ we know that $t \geq 2$ and then we view α^- as a formula of $\Gamma^-_{2,1}$); thus, (β^+, k) and (β^-, k) are the instances of p-WSAT$(\Gamma^+_{t,d})$ and p-WSAT$(\Gamma^-_{t,d})$, respectively, equivalent to $(\alpha, (\mathcal{X}_m)_{1 \leq m \leq k})$. □

Lemma 8. *For* $t, d \geq 1$*:*

1. p-WSAT$(\Gamma_{t,d}) \leq^{\text{efpt}} p$-PSAT$(\Gamma^+_{t,d})$ *for even* t.
2. p-WSAT$(\Gamma_{t,d}) \leq^{\text{efpt}} p$-PSAT$(\Gamma^-_{t,d})$ *for odd* t.

Proof: We only give the proof for odd t. The proof for even t is dual. Again the case $t = d = 1$ is trivial because p-WSAT$(\Gamma_{1,1})$ is in EPT. So we assume that $t + d \geq 3$.

Let (α, k) be an instance of p-WSAT$(\Gamma_{t,d})$ and let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be the set of variables of α. We introduce variables $X_{i,j}$ (for $1 \leq i \leq k$ and $1 \leq j \leq n$) and $Y_{i,j,j'}$ (for $1 \leq i < k$ and $1 \leq j < j' \leq n$) with the intended meaning

$X_{i,j}$: the ith variable set to TRUE is X_j

$Y_{i,j,j'}$: the ith variable set to TRUE is X_j and the $(i+1)$th is $X_{j'}$.

We group them into the sets $\mathcal{X}_i := \{X_{i,j} \mid 1 \leq j \leq n\}$ for $1 \leq i \leq k$ and $\mathcal{Y}_i := \{Y_{i,j,j'} \mid 1 \leq j < j' \leq n\}$ for $1 \leq i < k$. Note that an assignment satisfying $((\alpha_1 \wedge \ldots \wedge \alpha_{k-1}), (\mathcal{X}_i)_{1 \leq i \leq k}, (\mathcal{Y}_i)_{1 \leq i < k})$ and setting $X_{1,\ell_1}, \ldots, X_{k,\ell_k}$ to TRUE must set $Y_{1,\ell_1,\ell_2}, \ldots, Y_{k-1,\ell_{k-1},\ell_k}$ to TRUE, where

$$\alpha_i := \bigwedge_{1 \leq j \leq n} \Big(\bigwedge_{\substack{1 \leq j_1 < j_2 \leq n \\ j_1 \neq j}} (\neg X_{i,j} \vee \neg Y_{i,j_1,j_2}) \wedge \bigwedge_{\substack{1 \leq j_1 < j_2 \leq n \\ j_2 \neq j}} (\neg X_{i+1,j} \vee \neg Y_{i,j_1,j_2}) \Big)$$

for $1 \leq i < k$. Now we obtain α' from α by replacing negative literals $\neg X_j$ of α by

$$\bigwedge_{1 \leq i \leq k} \neg X_{i,j}$$

and positive literals X_j of α by the following formula expressing "that all intervals containing X_j are unchosen":

$$\bigwedge_{j<j'} \neg X_{1,j'} \wedge \bigwedge_{j'<j} \neg X_{k,j'} \wedge \bigwedge_{1\le i\le k-1} \bigwedge_{\substack{Y_{i,j_1,j_2}\in\mathcal{Y}_i\\ j_1<j<j_2}} \neg Y_{i,j_1,j_2}.$$

One easily verifies that $(\alpha' \wedge \alpha_1 \wedge \ldots \wedge \alpha_{k-1})$ is equivalent to a formula β in $\Gamma^-_{t,d}$ and that

$$\alpha \text{ is } k\text{-satisfiable} \iff (\beta, (\mathcal{X}_i)_{1\le i\le k}, (\mathcal{Y}_i)_{1\le i<k}) \in p\text{-PSAT}(\Gamma^-_{t,d}),$$

which gives the desired reduction (note that the parameter on the right hand side is $(2\cdot k-1) \in O(k)$). □

4.3. Model-checking problems. We start by recalling some definitions. A *(relational) vocabulary* τ is a finite set of relation symbols. Each relation symbol has an *arity*. The *arity* of τ is the maximum of the arities of the symbols in τ. A *structure* $\mathcal{A}$ of vocabulary τ, or τ*-structure* (or, simply structure), consists of a set A called the *universe*, and an interpretation $R^{\mathcal{A}} \subseteq A^r$ of each r-ary relation symbol $R \in \tau$. We synonymously write $\bar{a} \in R^{\mathcal{A}}$ or $R^{\mathcal{A}}\bar{a}$ to denote that the tuple $\bar{a} \in A^r$ belongs to the relation $R^{\mathcal{A}}$. For example, we view a *directed graph* as a structure $\mathcal{G} = (G, E^{\mathcal{G}})$, whose vocabulary consists of one binary relation symbol E. By definition, $\mathcal{G}$ is an (undirected) *graph*, if $E^{\mathcal{G}}$ is irreflexive and symmetric (that is, all graphs in this paper are simple and undirected).

We define the *size* of a τ-structure $\mathcal{A}$ to be the number

$$\|\mathcal{A}\| := |\tau| + |A| + \sum_{R\in\tau} \text{arity}(R) \cdot |R^{\mathcal{A}}|.$$

The overall length of a reasonable encoding of $\mathcal{A}$ (see [9] for details) only polynomially deviates from $\|\mathcal{A}\|$. For example, the size of a graph with n vertices and m edges is $O(n+m)$.

The class of all first-order formulas is denoted by FO. They are built up from atomic formulas using the usual boolean connectives and existential and universal quantifications. Recall that *atomic formulas* are formulas of the form $x = y$ or $Rx_1 \ldots x_r$, where $x, y, x_1, \ldots, x_r$ are variables and R is an r-ary relation symbol. The *size* $|\varphi|$ of a formula φ is the number of nodes of its syntax tree. The set of variables of the formula φ is denoted by var(φ). For $t \ge 1$, let Σ_t be the class of all FO-formulas of the form

$$\exists x_{11} \ldots \exists x_{1k_1} \forall x_{21} \ldots \forall x_{2k_2} \; \ldots \; Qx_{t1} \ldots Qx_{tk_t} \; \psi,$$

where $Q = \forall$ if t is even and $Q = \exists$ otherwise, and where ψ is quantifier-free. The class of Π_t-formulas is defined analogously starting with a block of universal quantifiers. Let $t, u \ge 1$. A formula φ is $\Sigma_{t,u}$, if it is Σ_t and all quantifier blocks after the leading existential block have length $\le u$. For example, a formula

$$\exists x_1 \ldots \exists x_k \forall y \exists z_1 \exists z_2 \psi,$$

where ψ is quantifier-free and k is arbitrary, is in $\Sigma_{3,2}$.

If $\mathcal{A}$ is a structure, $a_1, \ldots, a_n$ are elements of the universe A of $\mathcal{A}$, and $\varphi(x_1, \ldots, x_n)$ is a first-order formula whose free variables are among $x_1, \ldots, x_n$, then we write $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$ to denote that $\mathcal{A}$ satisfies φ if the variables $x_1, \ldots, x_n$ are interpreted by $a_1, \ldots, a_n$, respectively.

For a class C of structures and a class Φ of formulas, the *parameterized model-checking problem for structures in* C *and formulas in* Φ is defined as follows:

p-MC(C, Φ)	
Instance:	A structure $\mathcal{A}$ in C and a sentence φ in Φ.
Parameter:	$\lvert\varphi\rvert$.
Problem:	Decide whether $\mathcal{A}$ satisfies φ.

If C is the class of all structures, we denote p-MC(C, Φ) by p-MC(Φ).

The characterisation of the classes of the EW-hierarchy in terms of model-checking problems reads as follows:

Theorem 9. *For all $t \geq 2$ and $u \geq 1$, p-MC($\Sigma_{t,u}$) is complete for* EW$[t]$.

Proof: Let $t \geq 2$ and $u \geq 1$. The fpt-reduction from p-MC($\Sigma_{t,u}$) to p-WSAT($\Gamma_{t,1}$) given in [11] is in fact an efpt-reduction. Hence, p-MC($\Sigma_{t,u}$) $\in$ EW$[t]$.

For the hardness we restrict ourselves to even t (the proof for odd $t \geq 3$ being dual). By Lemma 8, it suffices to show that p-PSAT($\Gamma^+_{t,1}$) $\leq^{\text{efpt}}$ p-MC($\Sigma_{t,1}$). Let $(\alpha, (\mathcal{X}_m)_{1 \leq m \leq k})$ be an instance of p-PSAT($\Gamma^+_{t,1}$) with

$$\alpha = \bigwedge_{i_1 \in I} \bigvee_{i_2 \in I} \cdots \bigwedge_{i_1 \in I} \bigvee_{i_t \in I} X_{i_1,\ldots,i_t}.$$

The structure $\mathcal{A}$ has universe $A := \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k \cup I$. Furthermore, $\mathcal{A}$ contains unary relations for $\mathcal{X}_m$ $(1 \leq m \leq k)$ and for I, and a $t$ary relation $R^{\mathcal{A}}$ with

$$R^{\mathcal{A}} := \big\{(i_1, \ldots, i_{t-1}, X) \bigm| i_1, \ldots, i_{t-1} \in I \text{ and } X = X_{i_1,\ldots,i_{t-1},j} \text{ for some } j \in I\big\}.$$

Then $(\alpha, (\mathcal{X}_l)_{1 \leq l \leq k})$ belongs to p-PSAT($\Gamma^+_{t,1}$) if and only if $\mathcal{A} \models \varphi$, where

$$\varphi = \exists x_1 \ldots \exists x_k \big(\bigwedge_{1 \leq m \leq k} \mathcal{X}_m x_m \wedge \forall y_1 (I y_1 \to \exists y_2 (I y_2 \wedge \ldots \forall y_{t-1} (I y_{t-1} \to \bigvee_{1 \leq m \leq k} R y_1 \ldots y_{t-1} x_m) \ldots))).$$

Since φ is equivalent to a $\Sigma_{t,1}$-formula φ' with $\lvert\varphi'\rvert \in O(k)$, this gives the desired reduction. □

4.4. Fagin-definability. In [7, 10], two notions of definability of parameterized problems were introduced. The first is definability via model-checking problems, which we have considered in the previous section. The second is *Fagin-definability*, which we will consider now.

We will work with formulas with free *set variables*, usually denoted by capital letters X, Y, Z, which may simply be viewed as uninterpreted unary relation symbols. If $\mathcal{A}$ is a τ-structure with universe A, $B \subseteq A$, and $X \notin \tau$ is a set variable, then we write $(\mathcal{A}, X \leftarrow B)$ to denote the $\tau \cup \{X\}$-*expansion* of $\mathcal{A}$ in which X is interpreted as B. That is, $(\mathcal{A}, X \leftarrow B)$ is the $\tau \cup \{X\}$-structure with universe A, $S^{(\mathcal{A}, X \leftarrow B)} = S^{\mathcal{A}}$ for all $S \in \tau$, and $X^{(\mathcal{A}, X \leftarrow B)} = B$. We usually denote a formula φ with a free relation variable X by $\varphi(X)$ and then write $\mathcal{A} \models \varphi(B)$ instead of $(\mathcal{A}, X \leftarrow B) \models \varphi$.

For every first-order formula $\varphi(X)$ of vocabulary τ we let p-FD$_{\varphi(X)}$ be the following parameterized problem:

p-FD$_{\varphi(X)}$	
Instance:	A τ-structure $\mathcal{A}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Decide whether there is a subset S of A of cardinality k satisfying $\varphi(X)$ in $\mathcal{A}$, that is, with $\mathcal{A} \models \varphi(S)$.

We say that $\varphi(X)$ *Fagin-defines* the problem p-FD$_{\varphi(X)}$.

For all formulas φ, individual variables x, and set variables X, we write $(\exists x \in X)\varphi$ as an abbreviation of $\exists x(Xx \wedge \varphi)$ and $(\forall x \in X)\varphi$ as an abbreviation of $\forall x(Xx \rightarrow \varphi)$. For $t, d \geq 1$, we let $\Pi_{t/d}$ be the class of formulas $\varphi(X)$ of the form

$$\forall \bar{y}_1 \exists \bar{y}_2 \dots \forall \bar{y}_{t-1} (\exists z_1 \in X) \dots (\exists z_d \in X)\psi \tag{5}$$

in case t is even, and of the form

$$\forall \bar{y}_1 \exists \bar{y}_2 \dots \exists \bar{y}_{t-1} (\forall z_1 \in X) \dots (\forall z_d \in X)\psi \tag{6}$$

in case t is odd; here, $\bar{y}_1, \dots, \bar{y}_{t-1}$ denote finite sequences of variables and ψ is a quantifier-free formula not containing X. If all $\bar{y}_i$ have length 1, $\bar{y}_i = y_i$ and if $\psi = Ry_1 \dots y_{t-1} z_1 \dots z_d$ for a $(t-1)+d$-ary relation symbol R, then we speak of a *generic* $\Pi_{t/d}$-formula.

Often we implicitly will use the statements of the following two lemmas. One easily verifies that formulas of type (5) are monotone, formulas of type (6) are anti-monotone in the following sense:

Lemma 10. *Let $\varphi(X) \in \Pi_{t/d}$. Then, for a structure $\mathcal{A}$ and $k \in \mathbb{N}$ with $k \leq |A|$, we have:*

- *if t is even, then*

$$(\mathcal{A}, k) \in p\text{-FD}_{\varphi(X)} \quad \textit{iff} \quad (\mathcal{A}, \ell) \in p\text{-FD}_{\varphi(X)} \textit{ for some } \ell \leq k.$$

- *if t is odd, then*

$$(\mathcal{A}, k) \in p\text{-FD}_{\varphi(X)} \quad \textit{iff} \quad (\mathcal{A}, \ell) \in p\text{-FD}_{\varphi(X)} \textit{ for some } \ell \geq k.$$

By the next lemma it mostly suffices to consider generic formulas:

Lemma 11. *Let $t, d \geq 1$ and $\varphi_1(X), \varphi_2(X) \in \Pi_{t/d}$ with generic $\varphi_2(X)$. Then there is a constant $c \geq 1$ and a polynomial time algorithm associating with every structure $\mathcal{A}$ a structure $\mathcal{B}$ such that $|B| = |A|^c$ and such that for all $k \in \mathbb{N}$,*

$$(\mathcal{A}, k) \in p\text{-FD}_{\varphi_1(X)} \iff (\mathcal{B}, k) \in p\text{-FD}_{\varphi_2(X)}.$$

In particular, $p\text{-FD}_{\varphi_1(X)} \leq^{\text{efpt}} p\text{-FD}_{\varphi_2(X)}$.

Proof: Given $\varphi_1(X)$ as in (5) or as in (6), let c be the maximum length of the $\bar{y}_i$. By passing to an appropriate structure $\mathcal{B}_0$ with universe A^c, one can replace the blocks of quantifiers by single quantifiers, thus obtaining, say, in the case of even t, a formula $\varphi'(X) = \forall y_1 \exists y_2 \ldots \forall y_{t-1} (\exists z_1 \in X) \ldots (\exists z_d \in X) \psi'$ (compare the proof of Lemma 11 in [11] for details). Now, we can set $\mathcal{B} := (A^c, R^{\mathcal{B}})$, where $R^{\mathcal{B}}$ is the set ${\psi'}^{\mathcal{B}_0}$ of tuples satisfying ψ' in $\mathcal{B}_0$. □

Theorem 12. *Let $t \geq 2$. Then $\text{EW}[t]$ is the closure of the class of problems Fagin-defined by $\Pi_{t/1}$-formulas under ept-reductions.*

More precisely, for every $\Pi_{t/1}$-formula $\varphi(X)$ the problem $p\text{-FD}_{\varphi(X)}$ is contained in $\text{EW}[t]$, and for every generic $\Pi_{t/1}$-formula $\varphi(X)$, the problem $p\text{-FD}_{\varphi(X)}$ is complete for $\text{EW}[t]$.

This result is an immediate consequence of the following lemma taking $d = 1$.

Lemma 13. *For $t, d \geq 1$ and a generic $\Pi_{t/d}$-formula $\varphi(X)$:*

1. *There is a polynomial time algorithm associating with every structure $\mathcal{A}$ a propositional formula α such that for all $k \in \mathbb{N}$:*
   - *for even t, $\alpha \in \Gamma^+_{t,d}$ and $((\mathcal{A}, k) \in p\text{-FD}_{\varphi(X)} \iff (\alpha, k) \in p\text{-WSAT}(\Gamma^+_{t,d}))$;*
   - *for odd t, $\alpha \in \Gamma^-_{t,d}$ and $((\mathcal{A}, k) \in p\text{-FD}_{\varphi(X)} \iff (\alpha, k) \in p\text{-WSAT}(\Gamma^-_{t,d}))$;*
2. *There is a constant $c \geq 1$ and a polynomial time algorithm associating with every propositional formula $\alpha \in \Gamma^+_{t,d}$ (if t is even) and $\alpha \in \Gamma^-_{t,d}$ (if t is odd) a structure $\mathcal{A}$ with $|A| = |\alpha|^c$ such that for all $k \in \mathbb{N}$:*
   - *for even t, $((\alpha, k) \in p\text{-WSAT}(\Gamma^+_{t,d}) \iff (\mathcal{A}, k) \in p\text{-FD}_{\varphi(X)})$;*
   - *for odd t, $((\alpha, k) \in p\text{-WSAT}(\Gamma^-_{t,d}) \iff (\mathcal{A}, k) \in p\text{-FD}_{\varphi(X)})$.*

Proof: We present the proof for odd t, the proof for the even case being similar. Let $\varphi(X)$ be a generic $\Pi_{t/d}$-formula, that is,

$$\varphi(X) = \forall y_1 \exists y_2 \ldots \exists y_{t-1} (\forall z_1 \in X) \ldots (\forall z_d \in X) R y_1 \ldots y_{t-1} z_1 \ldots z_d.$$

For any structure $\mathcal{A}$ we let α be the $\Gamma^-_{t,d}$-formula

$$\alpha = \bigwedge_{a_1 \in A} \bigvee_{a_2 \in A} \cdots \bigvee_{a_{t-1} \in A} \bigwedge_{\substack{b_1, \ldots, b_d \in A, \\ \text{not } R^{\mathcal{A}} \bar{a} \bar{b}}} (\neg X_{b_1} \vee \ldots \vee \neg X_{b_d}).$$

Here, for $a \in A$, X_a is a propositional variable with the intended meaning "a is in the set satisfying $\varphi(X)$". Now, for every $k \in \mathbb{N}$, we have $((\mathcal{A}, k) \in p\text{-FD}_{\varphi(X)} \iff \alpha$ is k-satisfiable).

We turn to a proof of (2). Consider a formula $\alpha \in \Gamma^-_{t,d}$,

$$\alpha = \bigwedge_{i_1 \in I} \bigvee_{i_2 \in I} \cdots \bigvee_{i_{t-1} \in I} \bigwedge_{i_t \in I} (\neg X_{i_1,\ldots,i_t,1} \vee \ldots \vee \neg X_{i_1,\ldots,i_t,d}).$$

Let $\mathcal{X}$ be the set of variables of α. We set $n := |\alpha|$ and $A := \{1, \ldots, n\}$. We may assume that $I, \mathcal{X} \subseteq A$ and that the structure $\mathcal{A}$ with universe A has unary relations for these subsets. Moreover, $\mathcal{A}$ contains the $(t-1) + d$-ary relation

$$\begin{aligned} R^{\mathcal{A}} := \{(i_1, \ldots, i_{t-1}, X_1, \ldots, X_d) \mid i_1, \ldots, i_{t-1} \in I, X_1, \ldots, X_d \in \mathcal{X}, \\ \text{and for all } i_t \in I,\ \{X_{i_1,\ldots,i_t,1}, \ldots, X_{i_1,\ldots,i_t,d}\} \not\subseteq \{X_1, \ldots, X_d\}\}. \end{aligned}$$

Furthermore, we let $\psi(Y)$ be a $\Pi_{t/d}$-formula equivalent to

$$\begin{aligned} &\forall y_1 (I y_1 \to \\ &\quad \exists y_2 (I y_2 \wedge \ldots \\ &\quad\quad \exists y_{t-1} (I y_{t-1} \wedge \\ &\quad\quad\quad (\forall z_1 \in X) \ldots (\forall z_d \in X)(\mathcal{X} z_1 \wedge R y_1 \ldots y_{t-1} z_1 \ldots z_d)) \ldots)). \end{aligned}$$

Then,

$$(\alpha, k) \in p\text{-WSAT}(\Gamma^-_{t,1}) \iff (\mathcal{A}, k) \in \text{FD}_{\psi(Y)},$$

which, together with Lemma 11, proves our claim. □

5. Complete problems

In this section we show that two "non-logical" problems, the parameterized dominating set problem p-DS and the parameterized Vapnik-Chervonenkis problem p-VCDIM are complete for EW[2] and EW[3], respectively. In particular, this last result is remarkable, since in unbounded parameterized complexity theory p-VCDIM is W[1]-complete [4, 5].

A dominating set in a graph $\mathcal{G} = (G, E^{\mathcal{G}})$ is a subset $S \subseteq G$, such that all vertices $a \in G$ either are in S or are adjacent to some vertex in S (that is, $E^{\mathcal{G}}ab$ for some $b \in S$). Now, p-DS is the following problem:

p-DS	
Instance:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Decide whether $\mathcal{G}$ has a dominating set of size k.

The next theorem also contains a new, quite simple proof showing that p-DS is W[2]-complete.

Theorem 14. *p-DS $\equiv^{\text{efpt}}$ p-WSAT$(\Gamma^+_{2,1})$, so p-DS is EW[2]-complete.*

Proof: For p-DS $\le^{\text{efpt}}$ p-WSAT$(\Gamma^+_{2,1})$, let an instance of p-DS be given consisting of the graph $\mathcal{G} = (G, E^{\mathcal{G}})$ and $k \in \mathbb{N}$. We introduce a propositional variable X_a for every $a \in G$ and let α be the following $\Gamma^+_{2,1}$-formula:

$$\alpha = \bigwedge_{a \in G} \bigvee_{\substack{b \in G \\ b = a \text{ or } E^{\mathcal{G}}ab}} X_b.$$

Then, $((\mathcal{G}, k) \in p\text{-DS} \iff (\alpha, k) \in p\text{-WSAT}(\Gamma^+_{2,1}))$.

For p-WSAT$(\Gamma^+_{2,1}) \le^{\text{efpt}}$ p-DS consider an instance (α, k) of p-WSAT$(\Gamma^+_{2,1})$ with

$$\alpha = \bigwedge_{i \in I} \bigvee_{j \in J_i} X_{i,j}.$$

Letting $\mathcal{X}$ be the set of variables, we may assume that $|\mathcal{X}| \ge k$, that $I \cap \mathcal{X} = \emptyset$, and that no J_i is empty. Consider the graph

$$\mathcal{G} = (I \cup \mathcal{X}, E_1 \cup E_2),$$

where E_1 is the symmetric closure of $\{(i, X_{i,j}) \mid j \in J_i\}$ and E_2 just contains the edges that make $\mathcal{X}$ a clique. Then, $\mathcal{G}$ has a dominating set of size k if and only if α has a satisfying assignment of weight k. The direction from right to left is trivial: The set of variables set to TRUE in a satisfying assignment is a dominating set. For the other direction let a dominating set S of size k be given. If $S \subseteq \mathcal{X}$, then the assignment just setting the variables in S to TRUE satisfies α and we are done. Otherwise we can change S in order to achieve this form: Assume $i_0 \in I \cap S$. The vertex i_0 only has edges to the points in $\{X_{i_0,j} \mid j \in J_{i_0}\}$. Therefore, for every $j \in J_{i_0}$, the set $S_j = (S \setminus \{i_0\}) \cup \{X_{i_0,j}\}$ is a dominating set, too. If $X_{i_0,j} \notin S$ for some $j \in J_{i_0}$, then the corresponding S_j has cardinality k and we are done. If S already contains all $X_{i_0,j}$, then we add to $S \setminus \{i_0\}$ any variable $X \in \mathcal{X}$ not contained in S. □

We turn to the parameterized Vapnik-Chervonenkis problem. Let A be a finite set and $\mathcal{S} \subseteq \text{Pow}(A)$ a family of subsets of A. We say that $\mathcal{S}$ *shatters* a set $B \subseteq A$, if

$$B \cap \mathcal{S} := \{B \cap S \mid S \in \mathcal{S}\}$$

is the powerset $\text{Pow}(B)$ of B. The *Vapnik-Chervonenkis dimension of* $(A, \mathcal{S})$, denoted by VC$(A, \mathcal{S})$, is the maximum size of a set $B \subseteq A$ that is shattered by $\mathcal{S}$.

The parameterized Vapnik-Chervonenkis problem is defined as follows:

p-VCDIM	
Instance:	A finite set A, a family $\mathcal{S}$ of subsets of A, and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Decide whether VC$(A, \mathcal{S}) \ge k$.

We can represent a pair $(A, \mathcal{S})$, where A is a finite set and $\mathcal{S} \subseteq \mathrm{Pow}(A)$, as a structure $\mathcal{A}(A, \mathcal{S})$ of vocabulary $\{E, S\}$, where E is a binary and S a unary relation symbol: The universe of $\mathcal{A}(A, \mathcal{S})$ is $A \cup \mathcal{S}$, and the relations are

$$E^{\mathcal{A}(A,\mathcal{S})} := \{(a, S) \mid a \in A, S \in \mathcal{S}, a \in S\},$$
$$S^{\mathcal{A}(A,\mathcal{S})} := \mathcal{S}.$$

Then we can express that an instance has VC-dimension at least k by the Σ_1-formula

$$\varphi_k := \exists x_1 \ldots \exists x_k \exists y_1 \ldots \exists y_{2^k} \bigwedge_{I \subseteq \{1,\ldots,k\}} \bigvee_{j=1}^{2^k} \Big(\bigwedge_{i \in I} Ex_iy_j \wedge \bigwedge_{i \notin I} \neg Ex_iy_j \Big),$$

in the sense that ($\mathrm{VC}(A, \mathcal{S}) \geq k \iff \mathcal{A}(A, \mathcal{S}) \models \varphi_k$). This definition yields an fpt-reduction, defined by $(A, \mathcal{S}, k) \mapsto (\mathcal{A}(A, \mathcal{S}), \varphi_k)$, from p-VCDIM to p-MC(Σ_1), which implies that p-VCDIM is in W[1]. As a matter of fact, p-VCDIM is W[1]-complete [4, 5].

However, the reduction $(A, \mathcal{S}, k) \mapsto (\mathcal{A}(A, \mathcal{S}), \varphi_k)$ is not an ept-reduction, because the sentence φ_k and hence the parameter $|\varphi_k|$ of the model-checking problem is too large. The next lemma shows that with a slightly more complicated representation of the instances, p-VCDIM is $\Sigma_{3,1}$-definable by a formula of size linear in k.

Lemma 15. *For all $k \geq 1$ there is a $\Sigma_{3,1}$-sentence ψ_k, and for all sets A and families $\mathcal{S} \subseteq \mathrm{Pow}(A)$ a structure $\mathcal{B}(A, \mathcal{S}, k)$ such that*

$$\mathrm{VC}(A, \mathcal{S}) \geq k \iff \mathcal{B}(A, \mathcal{S}, k) \models \psi_k.$$

Furthermore, the mapping defined by $(A, \mathcal{S}, k) \mapsto (\mathcal{B}(A, \mathcal{S}, k), \psi_k)$ is an efpt-*reduction from p-*VCDIM *to p-*MC($\Sigma_{3,1}$).

Proof: Let $k \geq 1$ and $\tau_k := \{E, R_1, \ldots, R_k, P, Q, K\}$, where E is binary and $R_1, \ldots,$ R_k, P, Q, K are unary relation symbols. For every finite set A and $\mathcal{S} \subseteq \mathrm{Pow}(A)$, the structure $\mathcal{B} = \mathcal{B}(A, \mathcal{S}, k)$ is defined as follows:

- The universe of $\mathcal{B}$ is the set $B := A \cup \mathcal{S} \cup \mathrm{Pow}(\{1, \ldots, k\})$. Without loss of generality we may assume that the three sets A, $\mathcal{S}$, $\mathrm{Pow}(\{1, \ldots, k\})$ are disjoint.
- $E^{\mathcal{B}} := \big\{(a, S) \bigm| a \in A, S \in \mathcal{S}$ such that $a \in S\big\}$.
- For $1 \leq i \leq k$,

$$R_i^{\mathcal{B}} := \{L \mid L \in \mathrm{Pow}(\{1, \ldots, k\}), i \in L\}.$$

- $P^{\mathcal{B}} := A$, $Q^{\mathcal{B}} := \mathcal{S}$, $K^{\mathcal{B}} := \mathrm{Pow}(\{1, \ldots, k\})$.

We let

$$\psi_k := \exists x_1 \ldots \exists x_k \Big(\bigwedge_{1 \leq m \leq k} Px_m \wedge \forall y (Ky \to \exists z (Qz \wedge \bigwedge_{1 \leq m \leq k} (Ex_mz \leftrightarrow R_my))) \Big).$$

Then it is easy to verify that

$$\text{VC}(A, \mathcal{S}) \geq k \iff \mathcal{B} \models \psi_k.$$

Note that $\mathcal{B}(A, \mathcal{S}, k)$ and ψ_k can be computed from $A, \mathcal{S}, k$ in time polynomial in $|A| + |\mathcal{S}| + 2^k$ and that $|\psi_k|$ is linear in k. Thus the mapping $(A, \mathcal{S}, k) \mapsto \big(\mathcal{B}(A, \mathcal{S}, k), \psi_k\big)$ is an efpt-reduction. □

Theorem 16. *p-VCDIM is EW[3]-complete.*

Proof: p-VCDIM is contained in EW[3] by Lemma 15 and Theorem 9.

It remains to prove hardness. Our proof is based on Papadimitriou and Yannakakis's [15] proof that the (unparameterized) VC-dimension problem is hard for the class LOGNP (also see Section 6 of this paper). We shall prove that $p\text{-PSAT}(\Gamma^-_{3,1}) \leq^{\text{ept}} p\text{-VCDIM}$, thus obtaining the hardness of p-VCDIM by Lemma 8.

We view any Boolean matrix (that is, matrix with entries $0, 1$ only) $\mathcal{B} = (b_{ij})_{i \in I, j \in J}$ as a (partial) instance $(A(\mathcal{B}), \mathcal{S}(\mathcal{B}))$ of the Vapnik-Chervonenkis problem with $A(\mathcal{B}) := J$ and $\mathcal{S}(\mathcal{B}) := \{\{j \in J \mid b_{ij} = 1\} \mid i \in I\}$. Hence, the columns of $\mathcal{B}$ correspond to the elements and the rows to the subsets.

Let $X \subseteq J$ be a set of columns of $\mathcal{B}$. We say that a subset $Y \subseteq X$ is *realised* by a row $i \in I$ if for all $j \in X$ we have ($b_{ij} = 1 \iff j \in Y$). We say that X is *shattered (by $\mathcal{B}$)* if every subset Y of X is realised by some row of $\mathcal{B}$. Note that this is the case if and only if X is shattered by $\mathcal{S}(\mathcal{B})$.

Consider an instance $(\alpha, (\mathcal{X}_h)_{1 \leq h \leq k})$ of $p\text{-PSAT}(\Gamma^-_{3,1})$ with

$$\alpha = \bigwedge_{i \in I} \bigvee_{j \in J} \bigwedge_{\ell \in L} \neg X_{i,j,\ell}.$$

We may assume that $I = \{0, \ldots, n\}$ and that all $\mathcal{X}_h$ are ordered, so that we can speak of the sth variable in $\mathcal{X}_h$. We choose the minimal m such that $2^m > 2^k + |I| \cdot |J|$. We introduce a boolean matrix $\mathcal{B}$ such that for $k' := k + m + m$ we have

$$(\alpha, (\mathcal{X}_h)_{1 \leq h \leq k}) \in p\text{-PSAT}(\Gamma^-_{3,1}) \quad \iff \quad (A(\mathcal{B}), \mathcal{S}(\mathcal{B}), k') \in p\text{-VCDIM}. \quad (7)$$

The matrix $\mathcal{B}$ has three blocks of columns. The first block represents the selection of an assignment and is subdivided into k parts, the hth one has width $|\mathcal{X}_h|$. The second block has width m and will mainly contain the binary representations of natural numbers in I. The third block, the control part, also has width m. Let $\text{Bit}(s, a)$ denote the sth bit of the binary representation of $a \in \mathbb{N}$ (the 0th being the least important bit). Furthermore, for $a < 2^m$, let $\langle a \rangle = \text{Bit}(m-1, a) \ldots \text{Bit}(0, a)$ be the binary representation of a with m digits. And for $0 \leq a < 2^k$, let

$$\langle\langle a \rangle\rangle = \underbrace{\text{Bit}(0, a) \ldots \text{Bit}(0, a)}_{|\mathcal{X}_1| \text{ times}} \ldots \underbrace{\text{Bit}(k-1, a) \ldots \text{Bit}(k-1, a)}_{|\mathcal{X}_k| \text{ times}}.$$

The matrix $\mathcal{B}$ consists of the following rows:

(i) $\langle\langle e\rangle\rangle\ \langle i\rangle\ \langle s\rangle$ for all $0 \leq e < 2^k$, $0 \leq i < 2^m$, and $1 \leq s < 2^m$.

(ii) $\langle\langle e\rangle\rangle\ \langle i\rangle\ \langle 0\rangle$ for all $0 \leq e < 2^k$ and $n < i < 2^m$.

(iii) $\langle\langle e\rangle\rangle\ \langle i\rangle\ \langle 0\rangle$ for all $1 \leq e < 2^k$ and $0 \leq i \leq n$.

(iv) $w_1 \dots w_k\ \langle i\rangle\ \langle 0\rangle$ for all $0 \leq i \leq n$ and $j \in J$,
where every w_h has length $|\mathcal{X}_h|$ and depends on $(X_{i,j,\ell})_{\ell\in L}$: The sth position of w_h is 1 if and only if for some $\ell \in L$, the sth variable of $\mathcal{X}_h$ is $X_{i,j,\ell}$.

We call the rows defined in (i),(ii),(iii),(iv) rows *of type* (i),(ii),(iii),(iv), respectively.

Note that the matrix $\mathcal{B}$ and thus the instance $(A(\mathcal{B}), \mathcal{S}(\mathcal{B}), k')$ can be computed from $(\alpha, (\mathcal{X}_h)_{1\leq h\leq k})$ in time $2^{O(k)} \cdot |\alpha|^{O(1)}$ and that $k' \in O(k + \log|\alpha|)$. Thus to prove that the mapping

$$(\alpha, (\mathcal{X}_h)_{1\leq h\leq k}) \mapsto (X(\mathcal{B}), \mathcal{S}(\mathcal{B}), k')$$

is an ept-reduction, it only remains to prove (7).

Claim 1: Let X be a set of k' columns that is shattered by $\mathcal{B}$. Then X contains all columns of the last two blocks of $\mathcal{B}$ and exactly one column of each of the k parts of the first block.

Proof: Since the size of X is $k' = k + 2m$, it suffices to prove that X contains at most one column of each of the k parts of the first block.

If we restrict all rows to the first block, then at most $2^k + |I| \cdot |J| < 2^m$ rows occur. Therefore, the set X contains fewer than m columns of the first block. Since the length of the second block is m, the set X contains at least one column j from the last block. Note that a subset $Y \subseteq X$ that contains j can only be realised by a row of type (i).

Suppose for contradiction that X contains two columns j_1, j_2 of the same part of the first block. Since rows of type (i) are constant within each part of the first block, for every row i of type (i) we have ($b_{ij_1} = 1 \iff b_{ij_2} = 1$). Thus no row of $\mathcal{B}$ of type (i) and hence no row at all, can realise the subset $\{j_1, j\}$ of X, which contradicts the assumption that X is shattered by $\mathcal{B}$. This completes the proof of Claim 1.

Let us call a set X of k' columns of $\mathcal{B}$ that contains all columns of the last two blocks of $\mathcal{B}$ and exactly one column of each of the k parts of the first block *nice*. Each nice set of columns corresponds to an assignment to the variables of α that sets precisely the variables corresponding to the columns in X in the first block to TRUE.

Claim 2: Let X be a nice set of columns. Then X is shattered by $\mathcal{B}$ if and only if the assignment corresponding to X satisfies α.

Proof: Let $X_1 \in \mathcal{X}_1, \dots, X_k \in \mathcal{X}_k$ be the variables of α corresponding to the columns of the first block contained in X, and let b be the assignment corresponding to X, that is, the assignment that sets precisely $X_1, \dots, X_k$ to TRUE.

For the forward direction, suppose that X is shattered. To prove that b satisfies α, let $i \in I$. We shall prove that there is a $j \in J$ such that b satisfies $\bigwedge_{\ell\in L} \neg X_{i,j,\ell}$, or equivalently, that

$$X_1, \dots, X_k \notin \{X_{i,j,\ell} \mid \ell \in L\}. \tag{8}$$

Let $Y \subseteq X$ be the subset that contains no columns of the first and third block and precisely the columns of the second block corresponding to the positions of 1s in the binary representation of i. Since X is shattered, Y must be realised. Thus there is a row r of $\mathcal{B}$ of the form

$$w_1 \dots w_k \ \langle i \rangle \ \langle 0 \rangle,$$

in which all positions of the w_ps corresponding to columns in X must be 0. Since X contains exactly one column of each part of the first block, each w_p contains a position that is 0. Thus row r can only be of type (ii) or (iv). Since $i \in I$ and thus $0 \leq i \leq n$, row r must be of type (iv).

Suppose row r corresponds to $j \in J$. Recall that the columns of X in the first block correspond to variables $X_1, \dots, X_k$. By the definition of rows of type (iv), the position corresponding to variable X_p in row r is 1 if, and only if, $X_p = X_{i,j,\ell}$ for some $\ell \in L$. But by our choice of Y and of the row r, all these positions must be 0, thus $X_1, \dots, X_k \notin \{X_{i,j,\ell} \mid \ell \in L\}$. This proves (8).

For the backward direction, suppose that b satisfies α. Then for all i there is a j such that (8) holds. The row of type (iv) corresponding to i, j satisfies the subset Y of X defined as above. All other subsets are realised by rows of type (i)–(iii). This completes the proof of Claim 2.

Both claims together yield (7). □

6. The LOG-classes

In this section, we establish a connection between our bounded parameterized complexity theory and classical complexity. More specifically, we will be concerned with subclasses of NP[$\log^2 n$].

Consider the (classical) Vapnik-Chervonenkis problem

> VCDIM
> *Instance:* A finite set A, a family $\mathcal{S}$ of subsets of A, and $k \in \mathbb{N}$.
> *Problem:* Decide whether VC$(A, \mathcal{S}) \geq k$.

Since the power set of a set with s elements has cardinality 2^s, the VC-dimension of $(A, \mathcal{S})$ is at most log n where $n := |\mathcal{S}|$. Hence, there is a nondeterministic algorithm for VCDIM that uses $O(\log^2 n)$ nondeterministic bits.

We have a similar complexity for many parameterized problems if we restrict them to instances with parameter log n. Examples are the following problems LOG-CLIQUE and LOG-DS:

> LOG-CLIQUE
> *Instance:* A graph $\mathcal{G}$.
> *Problem:* Decide whether $\mathcal{G}$ has a clique of size log $|G|$.

> LOG-DS
> *Instance:* A graph $\mathcal{G}$.
> *Problem:* Decide whether $\mathcal{G}$ has a dominating set of size log $|G|$.

To analyse such problems, Papadimitriou and Yannakakis [15] introduced two new, "syntactically defined", subclasses LOGSNP and LOGNP of NP$[\log^2 n]$. Syntactically defined means that they are defined via logical complete problems reminiscent of our Fagin-defined problems. For every quantifier-free formula $\psi(w, x, \bar{y}, \bar{z})$ with tuples $\bar{y}$ and $\bar{z}$ of length p and q, respectively, and every $n \geq 1$, consider the formula

$$\varphi_n = \exists \bar{x} \in [n]^{\log n} \, \forall \bar{y} \in [n]^p \, \exists \bar{z} \in [n]^q \, \forall i \in [\log n] \, \psi(i, x_i, \bar{y}, \bar{z}). \tag{9}$$

We say that a structure $\mathcal{A}$ with universe $[n] = \{1, \ldots, n\}$ *satisfies* φ_n if there is a tuple $\bar{a} = (a_1, \ldots, a_{\log n}) \in [n]^{\log n}$ such that for all tuples $\bar{b} \in [n]^p$ there is a tuple $\bar{c} \in [n]^q$ such that for $1 \leq j \leq \lceil \log n \rceil$,

$$\mathcal{A} \models \psi(j, a_j, \bar{b}, \bar{c}).$$

For every quantifier-free formula $\psi(w, x, \bar{y}, \bar{z})$, consider the following problem:

Input:	A structure $\mathcal{A}$ with universe $[n]$.	(10)
Problem:	Decide whether $\mathcal{A}$ satisfies the formula φ_n of (9).	

A problem is defined to be in LOGNP if it is polynomial time reducible to the problem in (10) for some quantifier-free formula ψ.

The class LOGSNP is defined completely analogously, but with formulas φ_n of the form

$$\exists \bar{x} \in [n]^{\log n} \, \forall \bar{y} \in [n]^p \, \exists i \in [\log n] \, \psi(i, x_i, \bar{y}). \tag{11}$$

Papadimitriou and Yannakakis [15] proved that VCDIM is complete for LOGNP and LOG-DS is complete for LOGSNP, both under polynomial time reductions.

The syntactical definitions of the classes LOGNP and LOGSNP are similar to the "Fagin-definitions" of the classes EW[3] and EW[2], respectively, that are given in Theorem 12. We shall make this correspondence precise now.

For every first-order formula $\varphi(X)$ with the monadic second-order variable X we define the "logarithmic Fagin-definable" problem LOG-FD$_{\varphi(X)}$ by

LOG-FD$_{\varphi(X)}$	
Instance:	A structure $\mathcal{A}$.
Problem:	Decide whether there is a subset S of A of size $\log \lvert A \rvert$ with $\mathcal{A} \models \varphi(S)$.

Theorem 12 motivates the following definition:

Definition 17. For $t \geq 2$, let LOG$[t]$ be the class of problems that are polynomial time reducible to LOG-FD$_{\varphi(X)}$ for some $\Pi_{t/1}$-formula $\varphi(X)$.

Lemma 18. *Let $t \geq 2$. Then for every $\Pi_{t/1}$-formula $\varphi(X)$ and for every constant $c \geq 1$ the following problem is contained in* LOG$[t]$*:*

$\text{LOG}'\text{-FD}_{\varphi(X)}$ *Instance:* A structure $\mathcal{A}$ and a natural number $k \leq c \cdot \log |A|$. *Problem:* Decide whether there is a subset S of A of size k with $\mathcal{A} \models \varphi(S)$.

Proof: We only give the proof for even t. Odd $t \geq 3$ can be treated analogously. Let $\varphi(X)$ be a $\Pi_{t/1}$-formula. By Lemma 11 we may assume that

$$\varphi(X) = \forall y_1 \, \exists y_2 \ldots \forall y_{t-1} (\exists x \in X) R y_1 \ldots y_{t-1} x.$$

Let P, Q be unary relation symbols. Let $\varphi'(X)$ be a $\Pi_{t/1}$-formula equivalent to the following formula:

$$\begin{aligned} & (\forall y \in Q)(\exists x \in X) x = y \\ \wedge \, & (\forall y_1 \in P) \ldots (\forall y_{t-1} \in P)(\exists x \in X \cap P) R y_1 \ldots y_{t-1} x. \end{aligned}$$

$\varphi'(X)$ expresses that all elements of Q belong to X and that $\varphi(X)$ holds in the induced substructure with universe P.

For every $\{R\}$-structure $\mathcal{A}$ with universe A and all $\ell, m \geq 0$, let $\mathcal{A}_{\ell,m}$ be the $\{R, P, Q\}$-structure obtained from $\mathcal{A}$ by putting all elements of A into the relation P and then adding $\ell + m$ additional elements $a_1, \ldots, a_\ell, b_1, \ldots, b_m$ and putting $a_1, \ldots, a_\ell$ into Q.

Then for all $S \subseteq A$ and $S^+ \subseteq \{a_1, \ldots, a_\ell, b_1, \ldots, b_m\}$ with $\{a_1, \ldots, a_\ell\} \subseteq S^+$ we have

$$\mathcal{A} \models \varphi(S) \iff \mathcal{A}_{\ell,m} \models \varphi'(S \cup S^+). \tag{12}$$

Claim 1: For $0 \leq k \leq |A|$, there is an $S \subseteq A$ of size k such that $\mathcal{A} \models \varphi(S)$ if, and only if, there is an $S' \subseteq A \cup \{a_1, \ldots, a_\ell, b_1, \ldots, b_m\}$ of size $k + \ell$ such that $\mathcal{A}_{\ell,m} \models \varphi'(S')$.

Proof: The forward direction follows immediately from (12). For the backward direction, let $S' \subseteq A \cup \{a_1, \ldots, a_\ell, b_1, \ldots, b_m\}$ with $|S'| = k + \ell$ such that $\mathcal{A}_{\ell,m} \models \varphi'(S')$. Then $\{a_1, \ldots, a_\ell\} \subseteq S'$. Let $S_0 = S' \cap A$. By (12), we have $\mathcal{A} \models \varphi(S_0)$. Let $S \subseteq A$ be arbitrary with $S \supseteq S_0$ and $|S| = k$. By the monotonicity of $\varphi(X)$, we have $\mathcal{A} \models \varphi(S)$. This proves the claim.

Now we are ready to reduce $\text{LOG}'\text{-FD}_{\varphi(X)}$ to $\text{LOG-FD}_{\varphi'(X)}$. Let $\mathcal{A}$ be a τ-structure, $n = |A|$, and $k \leq c \cdot \log n$. We choose $\ell \geq 1$ minimal such that $2^{k+\ell} - \ell \geq n$. Let $m = 2^{k+\ell} - \ell - n$. Then

$$\log\,(n + \ell + m) = k + \ell.$$

Moreover, $\ell, m \leq 2^k \cdot n$, and ℓ, m can be computed in time polynomial in $2^k + n$. By Claim 1, the mapping $(\mathcal{A}, k) \mapsto \mathcal{A}_{\ell,m}$ gives the desired reduction. □

Proposition 19. LOG[2] = LOGSNP *and* LOG[3] = LOGNP.

Proof: Let $\varphi(X)$ be a $\Pi_{2/1}$-formula, say

$$\varphi(X) := \forall \bar{y} (\exists x \in X) \psi(x, \bar{y}),$$

where y is a p-tuple of variables. Then a structure $\mathcal{A}$ with universe $[n]$ is a positive instance of LOG-FD$_{\varphi(X)}$ if and only if it satisfies the formula

$$\exists \bar{x} \in [n]^{\log n} \, \forall \bar{y} \in [n]^p \, \exists i \in [\log n] \, \psi(x_i, \bar{y}),$$

which is of the form (11). To prove this, note that there is a subset $S \subseteq [n]$ of size $\log n$ such that $\mathcal{A} \models \varphi(S)$ if and only if there is a subset $S \subseteq [n]$ of size *at most* $\log n$ such that $\mathcal{A} \models \varphi(S)$. The latter is equivalent to the existence of a tuple $(a_1, \ldots, a_{\log n}) \in [n]^{\log n}$ such that $\mathcal{A} \models \varphi(\{a_1, \ldots, a_{\log n}\})$. Hence, LOG$[2] \subseteq$ LOGSNP.

For LOG$[3] \subseteq$ LOGNP, let $\varphi(X)$ be a $\Pi_{3/1}$-formula, say

$$\varphi(X) = \forall \bar{y} \exists \bar{z} (\forall x \in X) \psi(x, \bar{y}, \bar{z}),$$

where $\bar{y}$ is a p-tuple and $\bar{z}$ a q-tuple of variables. Then a structure $\mathcal{A}$ with universe $[n]$ is a positive instance of LOG-FD$_{\varphi(X)}$ if and only if it satisfies the formula

$$\exists \bar{x} \in [n]^{\log n} \, \forall \bar{y} u \in [n]^{p+1} \, \exists \bar{z} v \in [n]^{q+1} \, \forall i \in [\log n] \\ \big((x_i = u \to i = v) \wedge \psi(x_i, \bar{y}, \bar{z}) \big),$$

which is of the form (11). To see the equivalence, not that the formula

$$\forall u \in [n] \, \exists v \in [n] \, \forall i \in [\log n] \, (x_i = u \to i = v)$$

says that $x_1, \ldots, x_{\log n}$ are pairwise distinct.

To prove that LOGSNP $\subseteq$ LOG$[2]$, we show how to reduce a problem in LOGSNP to LOG$'$-FD$_{\varphi(X)}$ for some $\varphi \in \Pi_{2/1}$. So let $\psi(w, x, \bar{y})$, where $\bar{y} = (y_1, \ldots, y_p)$, be a quantifier-free formula (as in the definition of LOGSNP).

Let $\sigma = \{D, E_1, E_2, R\}$ be the vocabulary consisting of a unary relation symbol D, binary relation symbols E_1 and E_2, and a $(p+1)$-ary relation symbol R. For every structure $\mathcal{A}$ with universe $[n]$ and the same vocabulary as ψ we define a σ-structure $\mathcal{B}$ as follows: The universe of $\mathcal{B}$ is $B = A \times [k]$, where $k = \log n$, and the relations are defined by

$$\begin{aligned} D^{\mathcal{B}}(a, i) &\iff a = i, \\ E_1^{\mathcal{B}}(a, i)(a', i') &\iff a = a', \\ E_2^{\mathcal{B}}(a, i)(a', i') &\iff i = i', \\ R^{\mathcal{B}}(a, i)(b_1, j_1) \ldots (b_p, j_p) &\iff \mathcal{A} \models \psi(i, a, b_1, \ldots, b_p). \end{aligned}$$

The structure $\mathcal{B}$ can be obtained in polynomial time from $\mathcal{A}$. Let

$$\varphi(X) = \forall \bar{y} \forall u (\exists x \in X)((\neg Du \to Rx\bar{y}) \wedge (Du \to E_2 ux)).$$

Note $k \leq \log |B|$. We claim that $\mathcal{A}$ satisfies

$$\exists \bar{x} \in [n]^{\log n} \, \forall \bar{y} \in [n]^p \, \exists i \in [\log n] \, \psi(i, x_i, \bar{y}) \tag{13}$$

if and only if there is a set $S \subseteq B$ of size k such that $\mathcal{B} \models \varphi(S)$. This yields the desired reduction from the LOGSNP-problem defined by ψ to the problem $\text{LOG}'\text{-FD}_{\varphi(X)}$ (with $c = 1$).

To prove the claim, we first note that for every set

$$S = \{(a_i, j_i) \mid i \in [k]\} \subseteq B$$

such that $\mathcal{B} \models \varphi(S)$, the numbers $j_1, \ldots, j_k$ are pairwise distinct. This is ensured by the formula $\forall u(\exists x \in X)(Du \to E_2ux)$. Thus every set $S \subseteq B$ of size k such that $\mathcal{B} \models \varphi(S)$ consists of elements (a_i, i) for $i \in [k]$.

Furthermore, for every set

$$S = \{(a_i, i) \mid i \in [k]\},$$

if $\mathcal{B} \models \varphi(S)$ then for all $\bar{b} \in [n]^p$ and $(j_1, \ldots, j_p) \in [k]$ there exists an $i \in [k]$ such that $\big((a_i, i), (b_1, j_1), \ldots, (b_p, j_p)\big) \in R^{\mathcal{B}}$. By the definition of $R^{\mathcal{B}}$, this implies that for all $\bar{b} \in [n]^p$ there exists an $i \in [k]$ such that $\mathcal{A} \models \psi(i, a_i, \bar{b})$. Thus $\mathcal{A}$ satisfies the formula in (13).

For the converse direction, suppose that $(a_1, \ldots, a_k) \in [n]^k$ witnesses that $\mathcal{A}$ satisfies the formula in (13). Then by reversing the previous arguments, it is easy to see that

$$\mathcal{B} \models \varphi\big(\{(i, a_i) \mid i \in [k]\}\big).$$

It remains to prove that LOGNP $\subseteq$ LOG$[3]$. We proceed similarly. Let $\psi(w, x, \bar{y}, \bar{z})$ be a quantifier-free formula, $\mathcal{A}$ a structure with universe $[n]$, and $k = \log n$. We define a structure $\mathcal{B}$ exactly as above and let

$$\varphi(X) = \forall \bar{y} \forall u \exists \bar{z} \exists v (\forall x \in X)(Rx\bar{y}\bar{z} \wedge (E_2ux \to E_1vx)).$$

We claim that $\mathcal{A}$ satisfies

$$\exists \bar{x} \in [n]^{\log n}\, \forall \bar{y} \in [n]^p\, \exists \bar{z} \in [n]^q\, \forall i \in [\log n]\, \psi(i, x_i, \bar{y}, \bar{z})$$

if and only if there is a set $S \subseteq B$ of size $k = \log n$ such that $\mathcal{B} \models \varphi(S)$. This yields the desired reduction from the LOGNP-problem defined by ψ to the problem $\text{LOG}'\text{-FD}_{\varphi(X)}$ (with $c = 1$).

The crucial observation here is that for every set

$$S = \{(a_i, j_i) \mid i \in [k]\} \subseteq B$$

such that

$$(\mathcal{B}, X \leftarrow S) \models \forall u \exists v (\forall x \in X)(E_2ux \to E_1vx),$$

$j_1, \ldots, j_k$ are pairwise distinct. To see this, let S be such a set and suppose that $(a, i), (b, i) \in S$ for some $a \neq b \in [n]$ and $i \in [k]$. Let $c \in B$ such that

$$(\mathcal{B}, X \leftarrow S, u \leftarrow (i, i), v \leftarrow c) \models (\forall x \in X)(E_2ux \to E_1vx).$$

(This means that $(\forall x \in X)(E_2ux \to E_1vx)$ holds in $\mathcal{B}$ if X is interpreted by S, u is interpreted by (i, i), and v is interpreted by c.) Since both $((a, i), (i, i)) \in E_2^{\mathcal{B}}$ and

$((b,i),(i,i)) \in E_2^{\mathcal{B}}$, it follows that $((a,i),c) \in E_1^{\mathcal{B}}$ and $((b,i),c) \in E_1^{\mathcal{B}}$. This implies that $a = b$.

The proof is completed as for LOGSNP. $\square$

Recalling that LOG-DS is complete for LOGSNP and VCDIM is complete for LOGNP [15], we obtain:

Corollary 20. LOG-DS *is complete for* $\mathrm{LOG}[2]$ *and* VCDIM *is complete for* $\mathrm{LOG}[3]$.

For every class Γ of propositional formulas, we let

LOG-WSAT(Γ)
Instance: A propositional formula $\alpha \in \Gamma$.
Problem: Decide whether α is log $\lvert\alpha\rvert$-satisfiable.

Lemma 21. *For every* $t \geq 1$ *and every constant* $c \geq 1$ *the following problem is polynomial time reducible to* LOG-WSAT($\Gamma_{t,1}$)*:*

LOG$'$-WSAT($\Gamma_{t,1}$)
Instance: A formula $\alpha \in \Gamma_{t,1}$ and a natural number $k \leq c \cdot \log \lvert\alpha\rvert$.
Problem: Decide whether α has a satisfying assignment of weight k.

Proof: Let $t, c \geq 1$. Let $k \leq c \cdot \log \lvert\alpha\rvert$ and $\alpha = \bigwedge_{i=1}^{m} \beta_i \in \Gamma_{t,1}$. We shall define a formula $\alpha' \in \Gamma_{t,1}$ such that α has a satisfying assignment of weight k if and only if α' has a satisfying assignment of weight $\log \lvert\alpha'\rvert$.

Let $n = \lvert\alpha\rvert$, and $p \geq 1$ minimal such that $2^{k+p} - p \geq n$. Let $q = 2^{k+p} - p - n$. Then

$$\log\,(n + p + q) = k + p.$$

Moreover, $p, q \leq 2^k n$, and p, q can be computed in time polynomial in $2^k + n$.

Let $Y_1, \dots, Y_p$ be variables that do not occur in α and

$$\alpha' = \bigwedge_{i=1}^{m+p+q} \beta_i,$$

where

$$\beta_{m+j} = \begin{cases} Y_j & \text{if } 1 \leq j \leq p, \\ Y_1 & \text{if } p+1 \leq j \leq p+q. \end{cases}$$

Then $\lvert\alpha'\rvert = n + p + q$, and α' has a satisfying assignment of weight $k + p = \log \lvert\alpha'\rvert$ if and only if α has a satisfying assignment of weight k. $\square$

Theorem 22. *For every* $t \geq 2$, LOG-WSAT($\Gamma_{t,1}$) *is complete for* $\mathrm{LOG}[t]$ *under polynomial time reductions.*

Proof: LOG-WSAT$(\Gamma_{t,1}) \in \text{LOG}[t]$: By Theorem 12, there is a $\Pi_{t/1}$-formula $\varphi(X)$ such that p-WSAT$(\Gamma_{t,1})$ is efpt-reducible to p-FD$_{\varphi(X)}$. Let $\alpha \in \Gamma_{t,1}$ be given and set $k = \log|\alpha|$. The efpt-reduction produces in time $2^{c' \cdot k} \cdot p(|\alpha|)$, which is polynomial in $|\alpha|$, an equivalent instance $(\mathcal{A}, k')$ of p-FD$_{\varphi(X)}$, such that $k' \le d \cdot k$. In fact, the universe of the structure $\mathcal{A}$ produced by the reduction is at least as large as α (compare the proofs of Section 4). Hence, $k' \le c \cdot \log|A|$ for a suitable c. Therefore, $(\mathcal{A}, k')$ is an equivalent instance of LOG$'$-FD$_{\varphi(X)}$, too.

Completeness follows from the other direction of Theorem 12 and Lemma 21. □

Papadimitriou and Yannakakis [15] give a relatively complicated definition of LOG-CLIQUE by a formula of the form (11) and thus show that LOG-CLIQUE $\in \text{LOG}[2]$; here we get this result as an immediate consequence of Theorem 22 and Lemma 21 .

Corollary 23 ([15]). LOG-CLIQUE $\in \text{LOG}[2]$.

Corollary 24. $\text{LOG}[t] \subseteq \text{LOG}[t+1]$ *for all* $t \ge 2$.

Proof: Follows from Theorem 22 because $\Gamma_{t,1}$-formulas may be viewed as $\Gamma_{t+1,1}$-formulas. □

The last result of this section is a structural result that relates parameterized and classical complexity.

Theorem 25. *Let* $t \ge 2$. *Then,* $\text{EW}[t] = \text{EPT}$ *if and only if* $\text{LOG}[t] = \text{PTIME}$.

Proof: The forward direction is easy: If $\text{EW}[t] = \text{EPT}$, then p-WSAT$(\Gamma_{t,1})$ is solvable in time $2^{c \cdot k} \cdot p(n)$ for some constant c and polynomial p. Hence, LOG-WSAT$(\Gamma_{t,1})$ is solvable in time $2^{c \cdot \log n} \cdot p(n)$, which is polynomial. Now, the claim follows by Theorem 22.

For the backward direction, suppose that $\text{LOG}[t] = \text{PTIME}$ and let $\varphi(X)$ be a generic $\Pi_{t/1}$-formula. By Theorem 12 and Lemma 11, it suffices to show that p-FD$_{\varphi(X)}$ is in EPT.

Let $(\mathcal{A}, k)$ be an instance of p-FD$_{\varphi(X)}$. If $k \le \log|A|$, then we can solve the instance in polynomial time by our assumption that $\text{LOG}[t] = \text{PTIME}$ and by Lemma 18. To deal with instances where $k > \log|A|$, we define a $\Pi_{t/1}$-formula $\varphi'(X)$ and an efpt-reduction $(\mathcal{A}, k) \mapsto (\mathcal{A}', k')$ from p-FD$_{\varphi(X)}$ to p-FD$_{\varphi'(X)}$, which has the additional property that $k' \le \log|A'|$. Then we proceed as above.

The construction is very similar to the construction carried out in the proof of Lemma 18. Assume first that $t \ge 3$ is odd. Suppose that

$$\varphi(X) = \forall y_1 \exists y_2 \ldots \exists y_{t-1} (\forall x \in X) R y_1 \ldots y_{t-1} x$$

Let τ be the vocabulary of $\varphi(X)$ and P a unary relation symbol not contained in τ. Let $\varphi'(X)$ be a $\Pi_{t/1}$-formula equivalent to the following formula

$$(\forall x \in X) Px \wedge (\forall y_1 \in P)(\exists y_2 \in P) \ldots (\exists y_{t-1} \in P)(\forall x \in X) R y_1 \ldots y_{t-1} x.$$

$\varphi'(X)$ says that X is a subset of P and that $\varphi(X)$ holds in the induced substructure with universe P.

For every $\{R\}$-structure $\mathcal{A}$ with universe A and all $m \geq 0$, let $\mathcal{A}_m$ be the $\{R, P\}$-structure obtained from $\mathcal{A}$ by putting all elements of A into the relation P and then adding m new elements $a_1, \ldots, a_m$. Then for all $S \subseteq A$ we have

$$\mathcal{A} \models \varphi(S) \iff \mathcal{A}_m \models \varphi'(S).$$

Moreover, for all $S \subseteq A \cup \{a_1, \ldots, a_m\}$ with $\mathcal{A}_m \models \varphi'(S)$ we have $S \subseteq A$. Thus for all $k \geq 0$, there is an $S \subseteq A$ of size k such that $\mathcal{A} \models \varphi(S)$ if, and only if, there is an $S' \subseteq A \cup \{a_1, \ldots, a_m\}$ of size k such that $\mathcal{A}_m \models \varphi'(S')$.

Now we are ready to define the reduction. Let $\mathcal{A}$ be a τ-structure, $n = |A|$, and $k \geq 0$. Let $m = 2^k$, then

$$k \leq \log\,(m + n).$$

The mapping $(\mathcal{A}, k) \mapsto (\mathcal{A}_m, k)$ is the desired reduction.

In remains to give the deal with even $t \geq 2$. There is a slight problem because in this case we cannot say that X is a subset of P with a $\Pi_{t/1}$-formula. Nevertheless, we proceed more or less analogously. For

$$\varphi(X) = \forall \bar{y}_1 \, \exists \bar{y}_2 \ldots \forall \bar{y}_{t-1} (\exists x \in X) \psi(x, \bar{y}_1, \ldots, \bar{y}_{t-1}),$$

we let $\varphi'(X)$ be a $\Pi_{t/1}$-formula equivalent to the following formula:

$$(\forall \bar{y}_1 \in P) \ldots (\forall \bar{y}_{t-1} \in P)(\exists x \in X \cap P) \psi(x, \bar{y}_1, \ldots, \bar{y}_{t-1}).$$

For $m \geq 1$, we define $\mathcal{A}_m$ as above. Then for all $S \subseteq A$ and $S^+ \subseteq \{a_1, \ldots, a_m\}$ we have

$$\mathcal{A} \models \varphi(S) \iff \mathcal{A}_m \models \varphi'(S \cup S^+).$$

This implies that for all $k \leq |A|$, there is an $S \subseteq A$ of size k such that $\mathcal{A} \models \varphi(S)$ if, and only if, there is an $S' \subseteq A \cup \{a_1, \ldots, a_m\}$ of size k such that $\mathcal{A}_m \models \varphi'(S')$. To prove this, we use the monotonicity of $\varphi(X)$. The rest of the proof is analogous to the case of odd t. □

Corollary 26. *1.* EW[2] = EPT *if and only if* LOG-DS $\in$ PTIME.

2. EW[3] = EPT *if and only if* VCDIM $\in$ PTIME.

3. For all $t \geq 2$, EW[t] = EPT *if and only if* LOG-WSAT($\Gamma_{t,1}$) $\in$ PTIME.

7. Higher levels of intractability

We mentioned in Section 2 that p-MC(WORDS, FO) and even p-MC(WORDS, MSO) are in FPT but not in EPT (under the assumptions FPT $\neq$ AW[$*$] and P $\neq$ NP, respectively). In this section we analyse the $\mathfrak{E}$-parameterized complexity of these problems.

For a class of propositional formulas Γ the *alternating weighted satisfiability problem* AWSAT(Γ) is the following parameterized problem:

AWSat(Γ)	
Instance:	A formula $\alpha \in \Gamma$, a partition $(\mathcal{X}_m)_{1 \le m \le q}$ of its variables, and a sequence $(k_m)_{1 \le m \le q}$ of natural numbers.
Parameter:	$k_1 + \ldots + k_q$.
Problem:	Decide whether there is a size k_1 subset $\mathcal{Y}_1$ of $\mathcal{X}_1$ such that for every size k_2 subset $\mathcal{Y}_2$ of $\mathcal{X}_2$ there exists ... such that the truth value assignment only setting the variables in $\mathcal{Y}_1 \cup \ldots \cup \mathcal{Y}_q$ to TRUE satisfies α.

In unbounded parameterized complexity theory, the class AW[$*$] consists of all problems reducible to AWSAT$(\Gamma_{t,1})$ for some $t \ge 1$. Hence, we define:

$$\text{EAW}[*] := \bigcup_{t \ge 1} \left[\text{AWSAT}(\Gamma_{t,1})\right]^{\text{ept}}.$$

Similarly as in the unbounded theory, we have:

Proposition 27. AWSAT$(\Gamma_{1,2})$ *and, for* $t \ge 2$, AWSAT$(\Gamma_{t,1})$ *are complete for* EAW[$*$].

Proof: Since $\Gamma_{1,2} \subseteq \Gamma_{2,1} \subseteq \Gamma_{3,1} \subseteq \ldots$ up to obvious identifications, we have

$$\text{AWSAT}(\Gamma_{1,2}) \in \text{EAW}[*]$$

and it suffices to show that AWSAT$(\Gamma_{t,1}) \le^{\text{efpt}}$ AWSAT$(\Gamma_{1,2})$ for $t \ge 2$. The rest of the proof will rely on the following observation: Suppose we are given an instance

$$(\alpha, (\mathcal{X}_m)_{1 \le m \le q}, (k_m)_{1 \le m \le q})$$

of AWSAT(PROP) (PROP denotes the class of all propositional formulas), where

$$\alpha = \bigwedge_{i \in I} \bigvee_{j \in J} \alpha_{i,j}$$

with no restriction on the subformulas $\alpha_{i,j}$ and, say, with odd q. We let $\mathcal{X}_{q+1} = \{X_i \mid i \in I\}$ be a set of new variables and set

$$\alpha' = \bigvee_{i \in I} \bigvee_{j \in J} (X_i \wedge \alpha_{i,j})$$

and $k_{q+1} = 1$. Then $(\alpha', (\mathcal{X}_m)_{1 \le m \le q+1}, (k_m)_{1 \le m \le q+1})$ is an instance of AWSAT(PROP) equivalent to the original one (note that a universal quantifier is ranging over $\mathcal{X}_{q+1}$). This shows

$$\text{AWSAT}(\Gamma_{t+1,1}) \le^{\text{efpt}} \text{AWSAT}(\Delta_{t,1})$$

for $t \ge 2$, and

$$\text{AWSAT}(\Gamma_{2,1}) \le^{\text{efpt}} \text{AWSAT}(\Delta_{1,2}).$$

Dually one obtains

$$\text{AWSAT}(\Delta_{t+1,1}) \le^{\text{efpt}} \text{AWSAT}(\Gamma_{t,1})$$

for $t \geq 2$, and

$$\text{AWSAT}(\Delta_{2,1}) \leq^{\text{efpt}} \text{AWSAT}(\Gamma_{1,2}).$$

Composing such reductions we obtain $\text{AWSAT}(\Gamma_{t,1}) \leq^{\text{efpt}} \text{AWSAT}(\Gamma_{1,2})$ (since every formula in $\Delta_{1,2}$ is (equivalent to) a formula in $\Delta_{2,1}$). $\square$

We now turn to model-checking problems for the class of words. Given an alphabet Σ, we identify words $w \in \Sigma^*$ with structures $\mathcal{A}(w)$ as follows: As vocabulary we use $\tau_\Sigma := \{<, S\} \cup \{P_a \mid a \in \Sigma\}$, where $<$ is a binary relation symbol, S, the "successor", is a unary function symbol (only in this context, for easier formalizations, we consider vocabularies with a function symbol) and the P_a are unary relation symbols. The universe of the τ_Σ-structure $\mathcal{A}(w)$ is $\{1, \ldots, |w|\}$. The symbols $<$ and S are interpreted as the order relation and the successor function (with $S^{\mathcal{A}(w)}(|w|) = |w|$) on this subset of $\mathbb{N}$. A number i with $1 \leq i \leq |w|$ is in $P_a^{\mathcal{A}(w)}$ if and only if the ith letter of w is a. WORDS is the class of all structures that are words.

The following theorem is remarkable, since in unbounded parameterized complexity theory the problem p-MC(WORDS, FO) is in FPT whereas p-MC(FO) is AW[$*$]-complete. Moreover, it shows that

$$p\text{-MC(WORDS, FO)} \in \text{FPT}, \quad p\text{-MC(FO)} \leq^{\text{ept}} p\text{-MC(WORDS, FO)},$$
$$\text{and} \quad p\text{-MC(FO)} \notin \text{FPT},$$

in case FPT $\neq$ AW[$*$].

Theorem 28. *The following problems are complete for* EAW[$*$]*:*

1. p-MC(WORDS, FO).
2. p-MC(FO).

Proof: We start by showing $\text{AWSAT}(\Gamma_{1,2}) \leq^{\text{ept}} p$-MC(WORDS, FO). Let an instance $(\alpha, (\mathcal{X}_\ell)_{1\leq\ell\leq q}, (k_\ell)_{1\leq\ell\leq q})$ of $\text{AWSAT}(\Gamma_{1,2})$ be given, say, with even q. Then α has the form

$$\alpha = \bigwedge_{i \in I} (\alpha_{i,1} \vee \alpha_{i,2})$$

with literals $\alpha_{i,j}$. We set $\mathcal{X} = \bigcup_{1\leq\ell\leq q} \mathcal{X}_\ell$, say $\mathcal{X} = \{X_1, \ldots, X_{n_0}\}$. We choose m minimal with $n_0 < 2^m$.

We will construct an equivalent instance (w, φ) of p-MC(WORDS, FO), where w will be of the form $w = w_{\text{Var}} w_\alpha$ with w_{Var} representing the variables of α and w_α the formula α. The alphabet for w is

$$\Sigma = \{V_1, \ldots, V_q, +, -, 0, 1, \vee\}.$$

For $0 \leq n < 2^m$, we denote by $\langle n \rangle$ the binary representation of length m of n. The formula $\varphi_=(x, y)$ is such that, if the subwords of length m starting at x and y have the form $\langle n \rangle$ and $\langle n' \rangle$, then it states that $n = n'$:

$$\varphi_=(x, y) := \exists x_1 \dots \exists x_m \exists y_1 \dots \exists y_m \Big(x_1 = x \wedge y_1 = y \\ \wedge \bigwedge_{1 \le h < m} \big(x_{h+1} = S(x_h) \wedge y_{h+1} = S(y_h) \big) \\ \wedge \bigwedge_{1 \le h \le m} (P_0 x_h \leftrightarrow P_0 y_h) \Big).$$

Note that $|\varphi_=| = O(m) = O(\log |\alpha|)$. A variable $X_h \in \mathcal{X}_\ell$ is represented by the word $w_h = V_\ell \langle h \rangle$ and $w_{\text{Var}} = w_1 \dots w_{n_0}$ is the word representing all variables. For $1 \le \ell \le q$ the formula $\varphi_{\mathcal{Y}_\ell}(x_{\ell,1}, \dots, x_{\ell,k_\ell})$ expresses that $\bar{x}_\ell = x_{\ell,1} \dots x_{\ell,k_\ell}$ is an ascending sequence of positions carrying the letter V_ℓ; that is, that $\bar{x}_\ell$ corresponds to a subset of $\mathcal{X}_\ell$ of size k_ℓ:

$$\varphi_{\mathcal{Y}_\ell}(x_{\ell,1}, \dots, x_{\ell,k_\ell}) := \bigwedge_{1 \le i < k_\ell} x_{\ell,i} < x_{\ell,i+1} \ \wedge \bigwedge_{1 \le i \le k_\ell} V_\ell x_{\ell,i}.$$

The first-order formula φ we aim at will have the form

$$\exists \bar{x}_1 (\varphi_{\mathcal{Y}_1}(\bar{x}_1) \wedge \forall \bar{x}_2 (\varphi_{\mathcal{Y}_2}(\bar{x}_2) \rightarrow \quad \dots \quad \forall \bar{x}_q (\varphi_{\mathcal{Y}_q}(\bar{x}_q) \rightarrow \varphi') \dots)),$$

where φ' expresses that the truth assignment determined by $\bar{x} = x_{1,1} \dots x_{q,k_q}$ satisfies α. For this purpose a positive literal $\alpha_{i,j} = X_h$ is represented by the word $w_{i,j} = +\langle h \rangle$ and a negative literal $\alpha_{i,j} = \neg X_h$ by the word $w_{i,j} = -\langle h \rangle$. The formula $\varphi_L(\bar{x}, y)$ expresses that the literal starting at y is satisfied by the truth assignment $\bar{x}$:

$$\varphi_L(\bar{x}, y) := \exists z (\varphi_=(S(z), S(y)) \wedge \bigvee_{1 \le \ell \le q} (P_{V_\ell} z \wedge (P_+ y \leftrightarrow \bigvee_{1 \le i \le k_\ell} z = x_{\ell,i}))).$$

Read: If the variable of the literal y belongs to $\mathcal{X}_\ell$, then the literal is positive if and only if the variable is in $\bar{x}$. Finally, we represent α by the word w_α, which is a concatenation of all $\vee w_{i,1} w_{i,2}$ with $i \in I$. Then, setting

$$\varphi' := \forall z (P_\vee z \rightarrow (\varphi_L(\bar{x}, S(z)) \vee \varphi_L(\bar{x}, S^{m+2}(z))),$$

we have

$$\mathcal{A}(w_{\text{Var}} w_\alpha) \models \varphi \iff (\alpha, (\mathcal{X}_\ell)_{1 \le \ell \le q}, (k_\ell)_{1 \le \ell \le q}) \text{ belongs to AWSAT}(\Gamma_{1,2}).$$

The length $|\varphi|$ of φ can be bounded by $O(k + m) = O(k + \log |\alpha|)$, where $k := \sum_{\ell=1}^{q} k_\ell$ (is the old parameter). Therefore, this reduction is an ept-reduction but not an fpt-reduction.

Clearly, p-MC(FO)[WORDS] $\le^{\text{efpt}}$ p-MC(FO). Hence it remains to prove that

$$p\text{-MC(FO)} \in \text{EAW}[*].$$

In fact, we show: p-MC(FO) $\leq^{\text{efpt}}$ AWSAT$(\Gamma_{4,1})$. Let $(\mathcal{A}, \varphi)$ be an instance of p-MC(FO). In time $O(2^{|\varphi|})$ we can pass to a formula equivalent to φ of the form

$$\exists x_1 \forall x_2 \exists x_3 \ldots \forall x_q \bigwedge_{i \in I} \bigvee_{j \in J} \varphi_{i,j}$$

where the $\varphi_{i,j}$ are atomic or negated atomic formulas. We introduce variables $X_{m,a}$ for $1 \leq m \leq q$ and $a \in A$ with the intended meaning "the interpretation of x_m is a" and form the variable sets $\mathcal{X}_m = \{X_{m,a} \mid a \in A\}$ for $m = 1, \ldots, q$. Furthermore we set $k_m = 1$. Then, $k_1 + \ldots + k_q \in O(|\varphi|)$.

If $\varphi_{i,j}(x_{i_1}, \ldots, x_{i_r})$ is an atomic formula we let

$$\alpha_{i,j} = \bigvee_{\substack{a_1,\ldots,a_r \in A \\ \mathcal{A} \models \varphi_{i,j}(\bar{a})}} \bigwedge_{1 \leq h \leq r} X_{i_h, a_h}.$$

If on the other hand $\varphi_{i,j} = \neg\varphi'_{i,j}(x_{i_1}, \ldots, x_{i_r})$ with an atomic formula $\varphi'_{i,j}$, we let

$$\alpha_{i,j} = \bigwedge_{\substack{a_1,\ldots,a_r \in A \\ \mathcal{A} \models \varphi'_{i,j}(\bar{a})}} \bigvee_{1 \leq h \leq r} \neg X_{i_h, a_h}.$$

Note that in both cases we have $|\alpha_{i,j}| = O(||\mathcal{A}||)$, since the tuples of any relation of $\mathcal{A}$ are taken into consideration in the size $||\mathcal{A}||$ of $\mathcal{A}$. Finally, setting

$$\alpha = \bigwedge_{i \in I} \bigvee_{j \in J} \alpha_{i,j},$$

one easily verifies that

$$(\mathcal{A}, \varphi) \in p\text{-MC(FO)} \iff (\alpha, (\mathcal{X}_m)_{1 \leq m \leq q}, (k_m)_{1 \leq m \leq q}) \in \text{AWSAT}(\Gamma_{4,1}).$$

Altogether, the running time of this reduction is bounded by $2^{O(|\varphi|)} \cdot ||\mathcal{A}||$. □

We use the same technique to prove a result that helps to locate the complexity of p-MC(WORDS, MSO) in the EPT-world. For this purpose we introduce the *parameterized satisfiability problem* p-QBF *for quantified propositional logic*,

p-QBF	
Instance:	A sentence β of quantified propositional logic.
Parameter:	The quantifier alternation depth of β.
Problem:	Decide whether β is valid.

The reader familiar with the class para-NP will realize that p-QBF is hard for this class, since the first slice is hard for NP. So again, we have the remarkable fact that two problems, namely p-MC(WORDS, MSO) and p-QBF, with highly distinct FPT-complexities have the same EPT-complexity:

Theorem 29. *The following problems are equivalent under* ept*-reductions:*

1. p-QBF
2. p-MC(WORDS, MSO).
3. *For* $s \geq 1$, *the model-checking problem* p-MC(SO^s) *for the fragment* SO^s *of second-order logic* SO *consisting of all* SO*-formulas in which all quantified second-order variables have arity at most* s.

Proof: We only sketch the changes with respect to the proof of the preceding theorem.

p-QBF $\leq^{\text{ept}}$ p-MC(WORDS, MSO): Let β of alternation depth k be given. Using standard techniques, we may assume that β is in prenex normal form with k (maximal) blocks of quantifiers (without alternation), the first quantifier block being existential and that its quantifier-free part α is in conjunctive normal form with clauses of size exactly 3 (note that the new β can be obtained in polynomial time from the old one). For $1 \leq \ell \leq k$, let $\mathcal{X}_\ell$ be the set of variables of the i-th quantifier block and let α be the quantifier-free part of β. We aim at word w and a MSO-sentence φ such that ($\beta \in p$-QBF $\iff \mathcal{A}(w) \models \varphi$).

Again, the word w is of the form $w = w_{\text{Var}} w_\alpha$, where w_{Var} is as in the previous proof and w_α is an encoding of α similar to the one there. A truth assignment for the variables from $\mathcal{X}_\ell$ is a subset of those positions of the word labelled by V_ℓ and is represented in the formula φ by the monadic second-order variable X_ℓ:

$$\varphi = \exists X_1(X_1 \subseteq P_{V_1} \wedge \forall X_2(X_2 \subseteq P_{V_2} \to \dots \varphi_{\models})).$$

Here, $\varphi_{\models}(X_1, \dots, X_k)$ expresses that the truth assignment determined by the X_ℓ's satisfies α.

p-MC(SO^s) $\leq^{\text{efpt}}$ p-QBF: Let $(\mathcal{A}, \varphi)$ be an instance of p-MC(SO^s). We assume that φ is in prenex normal form and that $A = \{1, \dots, n\}$.

We introduce for every second-order variable Y of arity r (with $r \leq s$) propositional variables $X_{Y,i_1,\dots,i_r}$ for $1 \leq i_1, \dots, i_r \leq n$; the intended meaning of $X_{Y,i_1,\dots,i_r}$ is "$Y i_1 \dots i_r$ holds". And for a first-order variable y we introduce propositional variables $X_{y,i}$ for $1 \leq i \leq n$; $X_{y,i}$ says "y gets the value i". We inductively translate the subformulas ψ of φ into quantified propositional formulas β_ψ (and β_φ is the formula we aim at). We give the main steps:

1. If $\psi = R y_1 \dots y_r$ then
$$\beta_\psi = \bigvee_{(i_1,\dots,i_r) \in R^{\mathcal{A}}} \bigwedge_{1 \leq s \leq r} X_{y_s, i_s}.$$
2. If $\psi = Y y_1 \dots y_r$ then
$$\beta_\psi = \bigvee_{1 \leq i_1, \dots, i_r \leq n} (X_{Y,i_1,\dots,i_r} \wedge \bigwedge_{1 \leq s \leq r} X_{y_s, i_s}).$$
3. If $\psi = \exists y \psi'$ then
$$\beta_\psi = \exists X_{y,1} \dots \exists X_{y,n} \left(\bigvee_{1 \leq i \leq n} X_{y,i} \wedge \bigwedge_{1 \leq i < i' \leq n} \neg(X_{y,i} \wedge X_{y,i'}) \wedge \beta_{\psi'} \right).$$

4. If $\psi = \exists Y \psi'$ then

$$\beta_\psi = \exists X_{Y,1,\dots,1} \dots \exists X_{Y,i_1,\dots,i_r} \dots \exists X_{Y,n,\dots,n} \, \beta_{\psi'}.$$

Then, $\mathcal{A} \models \varphi$ if and only if β_φ is valid. Furthermore $|\beta_\varphi| = |\varphi| \cdot ||\mathcal{A}||^{O(1)}$ and the alternation depth of β_φ is the same as that of φ, hence bounded by $|\varphi|$. □

8. The EW-matrix

While for $t \geq 2$ and $d \geq 1$ the problems p-WSAT$(\Gamma_{t,d})$ and p-WSAT$(\Gamma_{t,1})$ are fpt-equivalent, they may not be ept-equivalent. In this section, we study the *EW-matrix* of classes

$$\text{EW}[t, d] := [p\text{-WSAT}(\Gamma_{t,d})]^{\text{ept}}$$

for $t, d \geq 1$. Clearly, EW$[1, 1] =$ EPT and we already know (cf. (3) and (4) in Section 4) that

$$\begin{aligned} \text{EW}[t, d] &= [p\text{-WSAT}(\Gamma^+_{t,d})]^{\text{ept}} \text{ for even } t \text{ and} \\ \text{EW}[t, d] &= [p\text{-WSAT}(\Gamma^-_{t,d})]^{\text{ept}} \text{ for odd } t. \end{aligned}$$

Also note that the classes of the EW-matrix can be linearly ordered by inclusion, because for all $t, d \geq 1$ we have

$$\text{EW}[t, d] \subseteq \text{EW}[t + 1, 1].$$

This follows immediately from the fact that every $\Gamma_{t,d}$-formula may be viewed as a $\Gamma_{t+1,1}$-formula.

In this section we give characterisations of the classes of the EW-matrix in terms of model-checking problems and in terms of Fagin-definable problems, thereby exemplifying a certain robustness of these classes.

So far we considered the model-checking problem parameterized by the length of the input formula; here, we need the same problem but now parameterized by the number of variables of the input formula, a parameterization already considered by Papadimitriou and Yannakakis [17]. For a class Φ of formulas, we define:

p-MC$_{\text{var}}(\Phi)$
Instance: A structure $\mathcal{A}$ and a sentence $\varphi \in \Phi$.
Parameter: $\lvert\text{var}(\varphi)\rvert$.
Problem: Decide whether $\mathcal{A} \models \varphi$.

Clearly, p-MC$(\Phi) \leq^{\text{efpt}} p$-MC$_{\text{var}}(\Phi)$.

Since in the new model-checking problem the formula φ is not the parameter, in order to put the quantifier-free part of φ into, say, disjunctive normal form, in general, we need time exponential in the input length (and not only exponential in the parameter as it is the case for the usual parameterization of the model-checking problem). This forces us to consider formulas that are already in the desired form.

Let $t, u, d \geq 1$. A $\Sigma_{t,u,d}$-formula is a $\Sigma_{t,u}$-formula

$$\varphi = \exists x_{11} \dots \exists x_{1k_1} \forall x_{21} \dots \forall x_{2k_2} \ \dots \ Qx_{t1} \dots Qx_{tk_t} \ \psi,$$

such that

- no atomic subformula contains more than d variables of the first block (that is, variables of the form $x_{1,i}$)
- in case t is odd, ψ is in disjunctive normal form and
- in case t is even, ψ is in conjunctive normal form.

Proposition 30. *For $t \geq 2$ and $u \geq 1$,*

$$p\text{-MC}_{\text{var}}(\Sigma_{t,u,1}) \equiv^{\text{efpt}} p\text{-MC}(\Sigma_{t,u}).$$

Proof: p-MC$(\Sigma_{t,u}) \leq^{\text{efpt}} p$-MC$_{\text{var}}(\Sigma_{t,u,1})$: In [11], an fpt-reduction $(\mathcal{A}, \varphi) \mapsto (\mathcal{A}', \varphi')$ from p-MC$(\Sigma_{t,u})$ to p-MC$(\Sigma_{t,u})$ is presented, where in φ' no atomic subformula contains more than one variable of the first block. This reduction is an efpt-reduction. Now, in time allowed by an efpt-reduction, the quantifier-free part of φ' is transformed into conjunctive or disjunctive normal form; thereby, the number of variables of the first block, the new parameter, does not change.

To obtain p-MC$_{\text{var}}(\Sigma_{t,u,1}) \leq^{\text{efpt}} p$-MC$(\Sigma_{t,u})$, we show that p-MC$_{\text{var}}(\Sigma_{t,u,1}) \leq^{\text{efpt}}$ p-MC$(\Sigma_{t,u+1})$; this suffices, since the fpt-reduction from p-MC$(\Sigma_{t,u})$ to p-MC$(\Sigma_{t,1})$ in [10] is an efpt-reduction, too.

Let $(\mathcal{A}, \varphi)$ be an instance of p-MC$_{\text{var}}(\Sigma_{t,u,1})$ and assume that t is even. Then,

$$\varphi = \exists x_1 \dots \exists x_\ell \forall \bar{y}_2 \exists \bar{y}_3 \dots \forall \bar{y}_t \bigwedge_{i \in I} \bigvee_{j \in J} \psi_{i,j}$$

with literals $\psi_{i,j}$. First we replace the conjunction $\bigwedge_{i \in I}$ in φ by a universal quantifier. For this purpose, we add to the vocabulary τ of $\mathcal{A}$ unary relation symbols R_i for $i \in I$ and consider an expansion $\mathcal{B} := (\mathcal{A}, (R_i^{\mathcal{B}})_{i \in I})$ of $\mathcal{A}$, where $(R_i^{\mathcal{B}})_{i \in I}$ is a partition of A into nonempty disjoint sets. Then,

$$\mathcal{A} \models \varphi \iff \mathcal{B} \models \exists x_1 \dots x_\ell \forall \bar{y}_2 \exists \bar{y}_3 \dots \forall \bar{y}_t \forall y \bigvee_{i \in I} \bigvee_{j \in J} (R_i y \wedge \psi_{i,j}).$$

Furthermore, we expand $\mathcal{B}$ to a structure $\mathcal{C}$ by adding for $m = 1, \dots, \ell$ a relation $T_m^{\mathcal{C}}$ of arity $1 + |\bar{y}_2| + \dots + |\bar{y}_t| + 1$ containing all tuples that satisfy at least one disjunct $(R_i y \wedge \psi_{i,j})$, where $\psi_{i,j}$ contains x_m and similarly, we add one relation $T^{\mathcal{C}}$ for those disjuncts that contain no variable from the first block. Then,

$$\mathcal{A} \models \varphi \iff \mathcal{C} \models \exists x_1 \dots x_\ell \forall \bar{y}_2 \exists \bar{y}_3 \dots \forall \bar{y}_t \forall y (T_1 x_1 \bar{y} \vee \dots \vee T_\ell x_\ell \bar{y} \vee T \bar{y})$$

where $\bar{y} = \bar{y}_2 \dots \bar{y}_t y$. Note that the formula on the right hand side is a $\Sigma_{t,u+1}$ -formula of length $O(\ell)$, so we have the desired reduction. □

The promised characterisation of the classes EW$[t, d]$ reads as follows:

Theorem 31. *1. p-MC$_{\text{var}}(\Sigma_{t,u,d})$ is complete for* EW$[t,d]$ *for* $t,d,u \geq 1$.

2. p-FD$_{\varphi(X)}$ is complete in EW$[t,d]$ *for every generic $\Pi_{t/d}$-formula $\varphi(X)$ and $t,d \geq 1$.*

Proof: The case $t = d = 1$ is trivial since EW$[1,1]$ = EPT and all problems are in EPT for $t = d = 1$. So let us assume that $t + d \geq 3$.

Part (2) follows from Theorem 5 and Lemma 13. We turn to part (1). By Proposition 30, we may assume that $d \geq 2$. We show that p-MC$_{\text{var}}(\Sigma_{t,1,d})$ is hard for EW$[t,d]$ by proving:

Claim: p-FD$_{\varphi(X)} \leq^{\text{efpt}} p$-MC$_{\text{var}}(\Sigma_{t,1,d})$ for every generic $\Pi_{t/d}$-formula $\varphi(X)$.

First, let t be odd. Then,

$$\varphi(X) = \forall y_1 \exists y_2 \ldots \exists y_{t-1} (\forall x_1 \in X) \ldots (\forall x_d \in X) R\bar{y}\,\bar{x}.$$

We set

$$\psi = \exists z_1 \ldots \exists z_k \forall y_1 \exists y_2 \ldots \exists y_{t-1} \Big(\bigwedge_{1 \leq i < j \leq k} z_i \neq z_j \wedge \bigwedge_{1 \leq m_1, \ldots, m_d \leq k} R\bar{y} z_{m_1} \ldots z_{m_d} \Big).$$

Then, ψ is a $\Sigma_{t,u,d}$-formula and for any instance $(\mathcal{A}, k)$ of p-FD$_{\varphi(X)}$ we have ($(\mathcal{A}, k) \in p$-FD$_{\varphi(X)} \iff (\mathcal{A}, \psi) \in p$-MC$_{\text{var}}(\Sigma_{t,u,d})$) and $|\text{var}(\psi)|) \in O(k)$.

If t is even and

$$\varphi(X) = \forall y_1 \exists y_2 \ldots \forall y_{t-1} (\exists x_1 \in X) \ldots (\exists x_d \in X) R\bar{y}\,\bar{x},$$

then we set

$$\psi = \exists z_1 \ldots \exists z_k \forall y_1 \exists y_2 \ldots \forall y_{t-1} \Big(\bigwedge_{1 \leq i < j \leq k} z_i \neq z_j \wedge \bigvee_{1 \leq m_1, \ldots, m_d \leq k} R\bar{y} z_{m_1} \ldots z_{m_d} \Big).$$

Again, ψ is a $\Sigma_{t,u,d}$-formula and the corresponding equivalence holds.

Claim: p-MC$_{\text{var}}(\Sigma_{t,u,d}) \in$ EW$[t,d]$.

Let an instance $(\mathcal{A}, \varphi)$ of p-MC$_{\text{var}}(\Sigma_{t,u,d})$ be given, say for even t, and with

$$\varphi = \exists x_1 \ldots \exists x_k \forall \bar{y}_2 \exists \bar{y}_3 \ldots \forall \bar{y}_t \bigwedge_{i \in I} \bigvee_{j \in J} \psi_{i,j}$$

with literals $\psi_{i,j} = \psi_{i,j}(\bar{y}, x_{g(i,j,1)}, \ldots, x_{g(i,j,d)})$ and $\bar{y} = \bar{y}_2 \ldots \bar{y}_t$. We introduce the propositional formula $\alpha \in \Gamma_{t,d}$ by

$$\bigwedge_{\bar{a}_2 \in A^{|\bar{y}_2|}} \bigvee_{\bar{a}_3 \in A^{|\bar{y}_3|}} \cdots \bigwedge_{\bar{a}_t \in A^{|\bar{y}_t|}} \bigwedge_{i \in I} \bigvee_{j \in J} \bigvee_{\substack{\bar{b} \in A^d \\ \mathcal{A} \models \psi_{i,j}(\bar{a}, \bar{b})}} (X_{g(i,j,i),b_1} \wedge \ldots \wedge X_{g(i,j,d),b_d})$$

(here $\bar{a} = \bar{a}_2 \ldots \bar{a}_t$). Moreover, let $\alpha_0 := \bigwedge_{s=1}^{k} \bigvee_{a \in A} X_{s,a}$. Clearly, $(\alpha \wedge \alpha_0)$ is equivalent to a formula in $\Gamma_{t,d}$ and one verifies that

$$(\mathcal{A}, \varphi) \in p\text{-MC}_{\text{var}}(\Sigma_{t,u,d}) \iff (\alpha \wedge \alpha_0) \text{ is } k\text{-satisfiable}$$

obtaining a reduction from p-MC$_{\text{var}}(\Sigma_{t,u,d})$ to p-WSAT$(\Gamma_{t,d})$. □

9. The first level of the EW-hierarchy

We turn to the class EW$[1]$, which we defined in Subsection 4.1 by the equality

$$\text{EW}[1] = [p\text{-WSAT}(\Gamma_{1,2})]^{\text{ept}}.$$

The following theorem contains characterisations of EW$[1]$ similar to those presented in Section 4 for the classes EW$[t]$ for $t \geq 2$. Moreover, it shows that the W$[1]$-completeness of the parameterized clique problem p-CLIQUE survives in EPT, where

> p-CLIQUE
> *Instance:* A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
> *Parameter:* k.
> *Problem:* Decide whether $\mathcal{G}$ has a clique of size k.

Theorem 32. *The following problems are complete for* EW$[1]$*:*

1. p-WSAT$(\Gamma^-_{1,2})$.
2. p-FD$_{\varphi(X)}$ *for every generic* $\Pi_{1/2}$*-formula* $\varphi(X)$.
3. p-MC$_{\text{var}}(\Sigma_{1,1,2})$.
4. p-CLIQUE.

Proof: The completeness of the first three items already have been proven by Lemma 7 and Lemma 8, by Lemma 13, and by Theorem 31, respectively. For item (4) we show that p-CLIQUE $\equiv^{\text{efpt}}$ p-WSAT$(\Gamma^-_{1,2})$.

p-CLIQUE $\leq^{\text{efpt}}$ p-WSAT$(\Gamma^-_{1,2})$: Let an instance of p-CLIQUE be given consisting of a graph $\mathcal{G} = (G, E^{\mathcal{G}})$ and the natural number k. We may assume that no vertex of $\mathcal{G}$ is adjacent to all other vertices (otherwise, we first reduce our instance to a graph with this property and some $\ell < k$). Consider the $\Gamma^-_{1,2}$-formula

$$\alpha = \bigwedge_{\substack{a,b \in G \\ a \neq b \text{ and } (a,b) \notin E^{\mathcal{G}}}} (\neg X_a \vee \neg X_b).$$

Here, X_a is a propositional variable "expressing that a is in the clique". Then, (α, k) belongs to p-WSAT if and only if $(\mathcal{G}, k)$ belongs to p-CLIQUE.

p-WSAT$(\Gamma^-_{1,2})$ $\leq^{\text{efpt}}$ p-CLIQUE: Observe that the formula we just obtained is "generic" for $\Gamma^-_{1,2}$, so we reverse our preceding translation: Let (α, k) be an instance of p-WSAT$(\Gamma^-_{1,2})$ with

$$\alpha = \bigwedge_{i \in I} (\neg X_{i,1} \vee \neg X_{i,2})$$

and with set $\mathcal{X}$ of variables. We may assume that $X_{i,1} \neq X_{i,2}$ (otherwise, we may simplify (α, k)). Then

$$((\mathcal{X}, \{(X,Y) \mid X \neq Y \text{ and } \{X,Y\} \neq \{X_{i,1}, X_{i,2}\} \text{ for all } i \in I\}), k)$$

is an instance of p-CLIQUE that is equivalent to (α, k). $\Box$

In unbounded parameterized complexity theory, we know that p-MC(Σ_1) is complete for W$[1]$. Hardness can be shown by the following fpt-reduction of p-CLIQUE to p-MC(Σ_1):

$$(\mathcal{G}, k) \in p\text{-CLIQUE} \quad \iff \quad (\mathcal{G}, \exists x_1 \ldots \exists x_k \bigwedge_{1 \le i < j \le k} E x_i x_j) \in p\text{-MC}(\Sigma_1). \tag{14}$$

Since the formula on the right hand side has size $O(k^2)$, this is not an ept-reduction. In fact, we only know:

Proposition 33. p-MC$(\Sigma_1) \in$ EW$[1]$.

We omit the proof which is implicit in the first part of the proof of Proposition 24 in [11].

Finally we remark:

Theorem 34. EW$[1] =$ EPT *if and only if* LOG-CLIQUE $\in$ PTIME.

Proof: If EW$[1] =$ EPT, then p-CLIQUE $\in$ EPT. Hence, we can determine whether a graph $\mathcal{G}$ has a clique of size log $|G|$ in time $2^{O(\log |G|)} \cdot p(\|\mathcal{G}\|)$ for some polynomial p and therefore, in polynomial time.

For the converse, we assume that LOG-CLIQUE is solvable in polynomial time and show that p-CLIQUE $\in$ EPT. Let $(\mathcal{G}, k)$ be an instance of p-CLIQUE with $\mathcal{G} = (G, E^{\mathcal{G}})$. We could solve it using the algorithm for LOG-CLIQUE, if $k = \log |G|$. We will establish this last condition through appropriate modifications.

We first construct an equivalent instance $(\mathcal{G}', k')$ with log $|G'| \le k'$. The graph $\mathcal{G}'$ is obtained by adding log $|G| + 1$ new vertices to $\mathcal{G}$ and connecting them with every other vertex, old or new. Then, $|G'| = |G| +$ log $|G| + 1$ and hence log $|G'| \le 1 +$ log $|G|$. For $k' := k +$ log $|G| + 1$, the graph $\mathcal{G}'$ has a k'-clique if and only if $\mathcal{G}$ has a k-clique. We have log $|G'| \le k'$.

Next we construct an equivalent instance $(\mathcal{G}'', k'')$ such that $k'' =$ log $|G''|$. For this purpose, we add $2^{k'} - |G'|$ new isolated vertices to $\mathcal{G}'$ and let $k'' = k'$. Now we apply the algorithm for LOG-CLIQUE to $\mathcal{G}''$. The running time is $\|\mathcal{G}''\|^{O(1)} = 2^{O(k')} = 2^{O(k)} \cdot |G|^{O(1)}$, hence we have an ept-algorithm solving p-CLIQUE. $\Box$

Theorem 25 and Theorem 34 suggest to define LOG$[1]$ as the closure of LOG-CLIQUE under polynomial time reductions.

10. Conclusions

We introduce a notion of bounded fixed-parameter tractability and develop a basic complexity theory for this notion of tractability. A particularly nice feature of this theory is its close connection with the classical complexity theory of problems that can be solved with log$^2 n$ nondeterministic bits.

By and large, the theory is well-behaved, but it is not as robust as one might wish. This is particularly true when it comes to the definition of the class EW$[1]$. The alternative notion of bounded fixed-parameter tractability is $\mathfrak{EXP}$-FPT, where the parameter dependence of a fixed-parameter tractable algorithm is bounded by $2^{\mathrm{poly}(n)}$. It is more robust and does not have these problems (on the other hand, it does not have the nice connection to the LOG-classes either). This theory is investigated in [19].

It remains an open problem whether the EW-matrix collapses to the EW-hierarchy, that is, whether for $t \geq 1$ and $d \geq 2$ we have $\mathrm{EW}[t, d] = \mathrm{EW}[t]$. It also remains open whether the parameterized model-checking problem for Σ_1-formulas is complete for the class $\mathrm{EW}[1]$.

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